BETTI NUMBERS OF STANLEY REISNER IDEALS

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This thesis compiles results in four related areas.

- **Jump Sequences of Edge Ideals:** Given a graph G on n vertices with edge ideal I_G, we introduce a new invariant $\text{Jump}(I_G)$ which describes the possible Betti tables of I_G. We show that the smallest k such that $\beta_{k,k+3}(I_G) \neq 0$ is bounded below in terms of smallest j such that $\beta_{j,j+2}(I_G) \neq 0$. In addition, we show that for ideals I_G such that $\beta_{2,4}(I_G) = 0$ and fewer than 11 vertices satisfy $\text{reg}(I_G) \leq 3$. We construct large classes of examples partially spanning the set of Betti tables of I_G with $\text{reg}(I_G) = k$.
- **Stabilization of Betti Tables:** Let R be a polynomial ring. Given a homogeneous ideal $I \subseteq R$ equigenerated in degree r, we show that the Betti tables of I^d stabilize into a fixed shape for all $d \geq D$ for some D.
- **Linear Quotients Ordering of Anticycle:** Let A_n be the anticycle graph on n vertices and P_n be the antipath graph on n vertices. We produce a linear quotients ordering on all powers of the edge ideal of the antipath $I_{P_n}^k$, and a linear quotients order on the second power of the edge ideal of the anticycle anticycle $I_{A_n}^2$.
- **Nerve Complexes of Graphs:** We examine the nerve complex $\mathcal{N}(G)$ of a graph G. We show that the Betti numbers of this complex encode spanning trees, matchings, genus, k-edge connectivity, and other invariants of G.

BIOGRAPHICAL SKETCH

Gwyn Whieldon is a commutative algebraist. Her undergraduate degree was in Mathematics and Physics from St. Mary's College of Maryland. She has lived in Ithaca, NY, on and off since the age of nineteen and will miss the summers there very much. Her non-mathematical interests include cycling, skiing, hiking, dancing salsa, and watching bad sci-fi movies. She is also an exceptionally poor gardener - but enjoys it immensely.

This thesis is dedicated to my incredibly supportive family, who have stood with me through everything (and more). To my mother and father, Althea and Charles, I owe much gratitude. Thanks for instilling in me an enjoyment of good debate and an intellectual restlessness. Thanks for many long Skype conversations during my time in graduate school about the escapades of the once-mine-now-yours cats, the difficulties inherent in teaching, the adventures you've had on the boat, and the envy-inducing stories about your garden.

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

BETTI NUMBERS OF SQUAREFREE MONOMIAL IDEALS

Let $R = \Bbbk[x_1, ..., x_n]$ and let M be an A-graded R-module. Let $\mathcal{F}$ be the augmented minimal graded free resolution of M,

$$\mathcal{F}: \qquad M \longleftarrow R \longleftarrow \bigoplus_{a \in A} R(-a)^{\beta_{0,a}} \longleftarrow \bigoplus_{a \in A} R(-a)^{\beta_{1,a}} \longleftarrow \cdots .$$

All $\mathbb{N}$-graded and $\mathbb{N}^k$-graded modules over R have finite resolutions by the Hilbert syzygy theorem. Hence we have for ideals $I \subset R$ and modules $M = R/I$

$$\begin{aligned}\mathcal{F}: \qquad R/I \longleftarrow R \longleftarrow \bigoplus_{a \in A} R(-a)^{\beta_{0,a}} \longleftarrow \bigoplus_{a \in A} R(-a)^{\beta_{1,a}} \longleftarrow \cdots \\ \cdots \longleftarrow \bigoplus_{a \in A} R(-a)^{\beta_{r-1,a}} \longleftarrow \bigoplus_{a \in A} R(-a)^{\beta_{r,a}} \longleftarrow 0,\end{aligned}$$

where the ranks of these modules, the Betti numbers $\beta_{i,a}(M)$ above, are an invariant of M.

In the most general case, it is an open question to describe concretely in terms of the combinatorial data of I effective bounds (either lower *or* upper) on the Betti numbers of these minimal resolutions. More generally, we wish to describe the behavior of the resolutions of powers I^k in terms of the data of ideal I and its resolution $\mathcal{F}$. This thesis focuses on four different results which constrain or determine the Betti numbers of certain classes of ideals or those of their powers.

Chapters 5 and 6 Preview:

Jump Sequences and Betti Numbers of Edge Ideals

The first set of results will focus on the Betti numbers of squarefree degree 2 ideals (also called *edge ideals* due to their correspondence with finite simple graphs.) We define a new pair of invariants of an edge ideal I_G, called the *jump sequence of* I_G and the *relative jump sequence of* I_G, respectively.

-	0	1	2	3	a_1+1	$\cdots$	a_2+1	$\cdots$	$a_{k-1}-1$	a_{k-1}	$a_{k-1}+1$
total:	1	β_1	β_2	$\cdots$	β_{a_1+1}	$\cdots$	β_{a_2+1}	$\cdots$	$\beta_{a_{k-1}-1}$	$\beta_{a_{k-1}}$	$\cdots$
0:	1	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	
1:	$\cdot$	$\beta_{1,2}$	$\beta_{2,3}$	$*$	$*$	$*$	$*$	$*$			
2:			a_1		β_{a_1+1,a_1+3}	$*$	$*$	$*$	$*$		
3:				a_2			β_{a_2+1,a_2+4}	$*$	$*$	$*$	$*$
$\vdots$									$*$	$*$	
k:						a_{k-1}					$\beta_{a_{k-1}+1,s}$

Figure 1.1: Betti Tables and Jump Sequences of I_G

If the resolution of the ideal I_G is of the form seen in Figure 1.1, with $\beta_{i,j}(I_G) = 0$ for all Betti numbers below the line, then the jump sequence of I is

$$\mathbf{a}_G = [k; a_1, ..., a_{k-1}].$$

The relative jump sequence is

$$\mathbf{r}_G = [k; a_1, a_2 - a_1, a_3 - a2, ..., a_{k-1} - a_{k-2}]$$
$$= [k; r_1, r_2, ..., r_{k-1}].$$

The jump sequence and relative jump sequence measure how quickly the degrees of the Betti numbers of I_G grow relative to the homological stage of the resolution. In particular, we can show that not all such sequences are achievable.

Knowing the first jump a_1 constrains the location of the next jump a_2. Specifically, we have the following:

Theorem 1.0.1 (Theorem 5.2.1)**.** Let I_G be an edge ideal with jump sequence a $=$ $[k; a_1, a_2, ..., a_{k-1}]$ and relative jump sequence $[k; r_1, r_2, ..., r_{k-1}]$. The following hold:

(i) $2a_1 \leq a_2$

(ii) $r_1 \leq r_2$.

In general, developing topological tools to deal with jump sequences of I_G and using such sequences to provide classes of examples achieving high regularity and projective dimension is the focus of Chapters 5 and 6. In the process, we construct explicit formulas for the Betti numbers of the Stanley-Reisner ideal of the join of two simplicial complexes:

Proposition 1.0.2 (Proposition 6.2.2)**.** Given two square-free ideals $I_{\Delta_1} \subset \mathbb{k}[x_1, ..., x_s]$ and $I_{\Delta_2} \subset \mathbb{k}[y_1, ..., y_t]$ with Stanley-Reisner complexes Δ_1 and Δ_2 respectively, the ideal $I_{\Delta_1 \uplus \Delta_2}$ given by

$$I_{\Delta_1 \uplus \Delta_2} = I_{\Delta_1} + I_{\Delta_2} + (x_i y_j : 1 \leq i \leq s, 1 \leq j \leq t)$$

is a square-free ideal with Stanley-Reisner complex $\Delta_1 \uplus \Delta_2$ and Betti numbers in the linear strand are

$$\begin{aligned}\beta_{i,i+1}(I_{\Delta_1 \cup \Delta_2}) = {} & \beta_{i,i+1}(I_{\Delta_1}) + \beta_{i,i+1}(I_{\Delta_2}) \\ & + \sum_{j=1}^{i} \left(\binom{m}{i-j+1} \beta_{j-1,j}(I_{\Delta_1}) + \binom{n}{j} \beta_{i-j,i-j+1}(I_{\Delta_2}) \right) \\ & + \binom{m+n}{i+1} - \binom{m}{i+1} - \binom{n}{i+1}.\end{aligned}$$

For terms in the nonlinear strands, we have for $s \geq 2$,

$$\begin{aligned}\beta_{i,i+s}(I_{\Delta_1 \cup \Delta_2}) = {} & \beta_{i,i+s}(I_{\Delta_1}) + \beta_{i,i+s}(I_{\Delta_2}) \\ & + \sum_{j=1}^{i+s-1} \left(\binom{m}{i-j+s} \beta_{j-s,j}(I_{\Delta_1}) + \binom{n}{j} \beta_{i-j,i-j+s}(I_{\Delta_2}) \right).\end{aligned}$$

Combining this with the example, due to Peeva and Nevo, of the edge ideal of the complement of the 1-skeleton of the 600-cell, we have the following:

Proposition 1.0.3 (Example 5.1.2)**.** There exist a family of graphs G_k such that indMatch$(G_k) = k$ and the regularity of I_{G_k} is

$$\text{reg} I_G = 4k - 1,$$

where indMatch(G) is the size of the largest set of edges which can be chosen in G such that the edges in the induced subgraph on those edges are pairwise disjoint. Specifically,

$$G_k = \underbrace{C_{600} * C_{600} * \cdots * C_{600}}_{k}$$

gives such a family.

The significance of indMatch(G) and its use in bounding regularity is discussed in depth in Chapter 5. Several main classes of edge ideals I_G achieving high regularity relative to the indMatch(G) are introduced.

Chapter 7 Preview:

Stabilization of Betti Tables of I^k

For an ideal $I \subseteq R = \Bbbk[x_1, ..., x_n]$, much work has been done on showing that regularity of I^d is a linear function in terms of d for high powers. The following theorem is a result of Cutkosky, Herzog and Trung:

Theorem 1.0.4 (Theorem 1.1 in [3])**.** Let I be an arbitrary homogeneous ideal. Let $r(I)$ denote the maximum degree of the homogeneous generators of I. The

(i) There is a number e such that reg$(I^d) \leq d \cdot r(I) + e$ for all $d \geq 1$.

(ii) reg(I^d) is a linear function for all d large enough.

They provide criteria for estimating this e in the case of an equigenerated ideal I, i.e. an ideal generated by homogeneous generators of the same degree. This result generalizes an earlier bound by Swanson giving the existence of an k such that

$$\text{reg}(I^d) \leq kd$$

for homogeneous ideals in [33].

Using techniques similar to those in [1] and [3], we produce here a stronger result on the resolutions of I^d.

Theorem 1.0.5 (Theorem 7.3.1, Betti Tables of Powers of Equigenerated Ideals)**.** Let $I = (f_0, f_1, ..., f_k) \subseteq \Bbbk[x_1, ..., x_n] = R$ be an equigenerated ideal of degree r. Then there exists a D such that for all $d > D$, we have

$$\beta_{i,j+rd}(I^d) \neq 0 \Longleftrightarrow \beta_{i,j+rD}(I^D) \neq 0.$$

This gives us that the shape of the Betti tables of powers of an ideal I is eventually fixed, translated down by the degree r of the ideal. It is unfortunately not the case that this guarantees that powers of our ideals I^d will be linear if they are linear for some I^D with $D < d$. See Example 7.0.6 for a related counterexample.

We also provide an upper bound for the Betti numbers of powers of an equigenerated ideal I in terms of the Betti numbers of the Rees ideal of I as follows.

Theorem 1.0.6. Let $I = (f_0, f_1, ..., f_k) \subseteq R = \Bbbk[x_1, ..., x_N]$ with f_i homogeneous of degree r. Let $\mathcal{R}(I)$ be the Rees algebra of I in ring $S = \Bbbk[x_1, ..., x_N, w_0, ..., w_k]$ with bigrading $\deg(x_i) = (1, 0)$ and $\deg(w_i) = (0, 1)$. Then

$$\beta_{i,j+rd}(I^d) \leq \sum_{m=0}^{d} \binom{d+k-m}{d-m} \beta_{i,(j,m)}(\mathcal{R}(I))$$

holds for all i, j, d.

The proof follows from a careful examination of the restriction of a minimal resolution of $\mathcal{R}(I)$ to bidegrees $(*, d)$.

We call the smallest such D for which Theorem 7.3.1 holds the *stabilization index Stab*(I) *of* I. A conjecture giving Stab(I) explicitly in terms of combinatorial data of I is presented.

Definition 1.0.7. Let I be a homogeneous equigenerated in polynomial ring R. Let Stab(I) be the smallest D such that for all $d \geq D$,

$$\beta_{i,j+rd}(I^d) \neq 0 \Longleftrightarrow \beta_{i,j+rD}(I^D) \neq 0.$$

For edge ideals I_G we conjecture Stab(I_G) explicitly. Areas of future research include proving this conjecture and producing Stab(I) for other classes of ideals. While Stab(I) is bounded in the proof of Theorem 7.3.1, we are interested in finding sharper upper bounds for $\text{Stab}(I)$, as found for the stabilization of regularity of I in [6].

Chapters 8 Preview:

Linear Quotients Ordering on the Anticycle

This is joint work with A. Hoefel.

Finding linear resolutions or linear quotients of powers of an ideal I^d is difficult, even in fairly simple cases. However, in the case of monomial ideals generated in degree 2 it is known that having a linear resolution of the ideal I is equivalent to having a linear resolution of all of its powers, via the following theorem due to Herzog, Hibi, and Zheng:

Theorem 1.0.8 (Theorem 3.2 in [18])**.** Let I be a monomial ideal generated in degree 2. The following conditions are equivalent:

(a) I has a linear resolution;

(b) I has linear quotients;

(c) Each power of I has a linear resolution.

Recall that an ideal I is said to have linear quotients if there exists some ordering of the generators of I, $(f_1, f_2, ..., f_k)$ such that $Q_i = (f_1, ..., f_{i-1}) : f_i$ is generated by linear forms for $i = 2, ..., k$.

The condition in Theorem 1.0.8 that I be equigenerated in degree 2 cannot be relaxed. There exists an ideal I generated in degree 3 with linear quotients, for which I^2 has a nonlinear resolution. This example, due to Sturmfels, has no linear quotients under any ordering of the generators.

Example 1.0.9. [Theorem 1.1 in [32]] Set

$$I = (def, cef, cdf, cde, bef, bcd, acf, ade) \subseteq \Bbbk[a, b, c, d, e, f].$$

The ideal I has a linear resolution and linear quotients with respect to the ordering given above, but I^2 fails to be linear. The resolutions are seen in Figure 1.2.

I

-	0	1	2	3
total:	1	8	11	4
0:	1	·	·	·
1:	·	·	·	·
2:	·	8	11	4

I^2

-	0	1	2	3	4	5	6
total:	1	36	85	79	38	10	1
0:	1	·	·	·	·	·	·
1:	·	·	·	·	·	·	·
2:	·	·	·	·	·	·	·
3:	·	·	·	·	·	·	·
4:	·	·	·	·	·	·	·
5:	·	36	84	75	32	6	·
6:	·	·	1	4	6	4	1

Figure 1.2: Ideals I with linear resolution and I^2 with a nonlinear resolution.

The edge ideal I_{A_n} of the anticycle does *not* have linear quotients, but it has been shown that the the square $I_{A_n}^2$ has a linear resolution, in [29] and [28]. Although orderings on the generators of $I_{A_n}^2$ which come from monomial term orderings fail to produce a linear quotients ordering, a linear quotients ordering does exist. We produce here a family of such linear quotients orderings on the generators of $I_{A_n}^2$ for all n, making use of the structure of the graph I_{A_n}.

To construct this linear quotients ordering, we begin by exhibiting a linear quotients ordering for all powers of the antipath. Following this, we decompose the generators of $I_{A_n}^2$ into three main types. In each type, we order the generators and demonstrate these orderings give us linear quotients. With this ordering in hand, we recover Peeva and Nevo's result that the square of the edge ideal of the anticycle has a linear resolution.

Chapters 9 Preview:

Nerve Complexes of Graphs

Definition 1.0.10. Let Δ be a simplicial complex on vertex set $V = \{v_1, ..., v_n\}$ with facet set $\{F_1, ..., F_k\}$. Then the *nerve complex or nerve* $\mathcal{N}(\Delta)$ *of* Δ is the simplicial complex on vertex set $\{w_1, ..., w_k\}$ with faces

$$\sigma = \{w_{i_1}, ..., w_{i_d} : F_{i_1} \cap \cdots \cap F_{i_d} \neq \emptyset\}.$$

A systematic study of nerves of simplicial complexes can be found in[14]. We enumerate certain subgraphs of G via the Betti numbers of the resolution of $\Bbbk[\mathcal{N}(G)]$, the Stanley-Reisner ring of $\mathcal{N}(G)$. Additionally, we characterize the generating sets and regularity of the Stanley-Reisner ideals $I_{\mathcal{N}(G)}$, as well as construct their Alexander duals and Hilbert functions.

Of interest are the subgraphs of G which are maximal trees. The set of all spanning trees $T(G)$ are given as the set of vanishing multigraded Betti numbers in a fixed degree.

Theorem 1.0.11 (Enumeration of Spanning Trees, Theorem 9.3.1)**.** Let G be a graph on vertex set $\{x_1, ..., x_n\}$ with edges $\{e_1, ..., e_k\}$. Then the set of spanning trees $T(G)$ of G is given by

$$T(G) = \{\{e_{i_1}, ..., e_{i_{n-1}}\} : \beta_{n-3,\mathbf{m}}(\Bbbk[\mathcal{N}(G)]) = 0, \quad \mathbf{m} = e_{i_1} \cdots e_{i_{n-1}}\}.$$

We also enumerate via $\mathcal{N}(G)$ all subgraphs of minimal cycles $\text{MinCycle}_k(G)$, Hamiltonian cycles $H(G)$, maximal and minimal vertex degrees in G, matchings $M_k(G)$ of size k, k-edge-connectivity $e_k(G)$ of G , and the Tutte polynomial $T_G(x, y)$ of G.

CHAPTER 2

BACKGROUND AND DEFINITIONS

2.1 Simplicial Complexes

Simplicial complex will be used here to mean any finite combinatorial simplicial complex, independent of geometric realization.

Definition 2.1.1. Let V denote the (finite) set $\{v_1, v_2, ..., v_n\}$ and 2^V denote the set of all subsets of V. A *simplicial complex* Δ *on* V is a collection of subsets $\sigma \in 2^V$ such that

(i) $\emptyset \in \Delta$,

(ii) $\{v_i\} \in \Delta$ for all $i = 1, ...n$ (all vertices of V in Δ), and

(iii) if $\tau \subset \sigma$ and $\sigma \in \Delta$, then $\tau \in \Delta$ (downward closure).

A subset $\sigma \in \Delta$ is called a *face of* Δ and a maximal face is called a *facet of* Δ. If $|\sigma| = d + 1$, we say that $\dim(\sigma) = d$.

We standardize our notation for the boundary of a face of Δ.

Definition 2.1.2 (Boundary)**.** Let $\sigma \in \Delta$ be a k-dim'l face of Δ, with

$$\sigma = \{v_{i_1}, ..., v_{i_{k+1}}\}.$$

The (nonoriented) *boundary of* σ is the set of $(k-1)$-dim'l faces

$$\partial\sigma := \{\sigma_j : \sigma_j = \{v_{i_1}, ..., \widehat{v_{i_j}}, ..., v_{i_{k+1}}\}, j = 1, ..., k+1\}$$

where $\widehat{v_{i_j}}$ denotes removal of vertex v_{i_j}.

2.1.1 (Multi-)Graded Modules and Ideals

All modules in this thesis will be positively $\mathbb{N}$-graded (*singly graded*), $\mathbb{N}^2$-graded (*bigraded*) or $\mathbb{N}^k$ (*finely graded or multigraded.*)

Definition 2.1.3. Let R be a ring and let A be a monoid with additive identity $0 \in A$. R is an *A-graded ring* if

$$R = \bigoplus_{a \in A} R_a$$

such that $R_i R_j \subseteq R_{i+j}$.

We extend this definition to graded modules.

Definition 2.1.4. M is an *A-graded R-module* if

$$M = \bigoplus_{j \in A} M_j$$

with $M_0 = R$ and for $i, j \in A$, we have $R_i M_j \subseteq M_{i+j}$. We say that $f \in M_i$ are *homogeneous elements of M*.

Given any element $f \in M$ in an A-graded R-module, we can decompose it into its homogeneous components $f = \sum_{a \in A} f_a$ such that $f_a \in M_a$. Ideals in R should interact naturally with this grading.

Definition 2.1.5. Let

$$I = \bigoplus_{a \in A} I_a \subset R$$

be an ideal and let f be an element of I. If $f = \sum_{a \in A} f_a$ with $f_a \in I_a$ we say f_a is a *homogeneous part of f*. We say that I *is a homogeneous ideal* if for every element $f = \sum_{a \in A} f_a \in I$, we have $f_a \in I$.

For $\mathbb{N}$-graded or $\mathbb{N}^k$-graded R-modules, we have graded maximal ideals

$$\mathfrak{m} = \bigoplus_{j \geq 1} M_j \quad \text{and} \quad \mathfrak{m} = \bigoplus_{\mathbf{a} \in \mathbb{N}^{\mathbf{n}}_{+}} M_{\mathbf{a}}$$

respectively, where $\mathbb{N}^n_+ = \{\mathbf{a} : \mathbf{a} = (a_1, ..., a_n) \neq \mathbf{0}\}$.

Example 2.1.6 (Running Example). Let $R = \Bbbk[x, y, z, w]$ be a polynomial ring, and $I = (x^2, y^2, xz - yw)$ be a homogeneous ideal. Then $M = R/I$ has homogeneous components M_i spanned as vector spaces over $\Bbbk$ by the following monomials:

$$\begin{aligned}
M_0 &= \text{span}\{1\} \\
M_1 &= \text{span}\{x, y, z, w\} \\
M_2 &= \text{span}\{xy, xz, xw, yz, z^2, zw, w^2\} \\
M_3 &= \text{span}\{xz^2, xzw, xw^2, yz^2, z^3, z^2w, zw^2, w^3\} \\
\vdots &= \quad \vdots \\
M_i &= \text{span}\{xS_{i-1}, yz^{i-1}, S_i\}
\end{aligned}$$

where $S = \Bbbk[z, w]$, with S_i the i^{th} homogeneous component of S.

We will also need some notation for altering the grading of our modules by shifting up or down by some monoid element $a \in A$.

Definition 2.1.7. We say that our module M is *twisted by* $a \in A$, denoted $M(a)$, if for $b \in A$, the b^{th} homogeneous component of $M(a)$ is given by

$$M(a)_b = M_{a+b}.$$

Example 2.1.8 (Running Example). Let $R = \Bbbk[x, y, z, w]$, $I = (x^2, y^2, xz - yw)$ and $M = R/I$. Then twisting M by 2, we have

$$M(2)_0 = \text{span}\{xy, xz, xw, yz, z^2, zw, w^2\} = M_2$$

as computed in the previous example.

Given two A-graded R-modules, we would like to talk about graded maps between them.

Definition 2.1.9. Let M, N be graded modules over A. We say that $\phi : M \to N$ is a graded map of degree $a \in A$ if for each graded piece M_b of M, we have

$$\phi(M_b) \subseteq N_{a+b}.$$

The most important special case we will consider is ϕ of degree zero as a graded map of modules, so $\phi(M_a) \subseteq N_a$ for all $a \in A$.

Example 2.1.10. The map taking us from a polynomial ring R to its quotient by a homogeneous ideal is a degree zero map.

2.1.2 Free Resolutions and Betti Numbers of M

This thesis focuses on computing invariants of the free resolution of the module M, which is a chain complex of modules measuring precisely how far a module is from free. We approximate the module by taking the module of relations on the generators of M, called *syzygies of M*, then the module of relations on those relations, etc. The i^{th} module F_i in this list is called the *i^{th} syzygy module of M.*

Definition 2.1.11. Let M be a module over a ring R. We say that a chain complex $\mathcal{F}$ of R-modules F_i,

$$\mathcal{F} : F_0 \longleftarrow F_1 \longleftarrow F_2 \longleftarrow \cdots \longleftarrow F_i \longleftarrow F_{i+1} \longleftarrow \cdots$$

is a free resolution of M if the complex is exact except at the leftmost position, where M is the cokernel of the final map. The *augmented free resolution* $\mathcal{F}$ is the same as above, but adding a final map to M.

$$\mathcal{F} : M \longleftarrow F_0 \longleftarrow F_1 \longleftarrow F_2 \longleftarrow \cdots \longleftarrow F_i \longleftarrow F_{i+1} \longleftarrow \cdots$$

All standardly $\mathbb{N}$-graded or $\mathbb{N}^k$-graded R-modules M, with R a polynomial ring, have a finite free resolution by the Hilbert Sygyzy Theorem (Theorem 1.13 in [5].) Such resolutions are a standard tool for investigating the structure of modules. For graded resolutions of graded modules, we require all maps to be of degree zero.

Definition 2.1.12. Let M be an A-graded R-module. The chain complex $\mathcal{F}$ is a graded free resolution of M,

$$\mathcal{F}: \quad M \longleftarrow \bigoplus_j S(-j)^{\beta_{0,j}} \longleftarrow \bigoplus_j S(-j)^{\beta_{1,j}} \longleftarrow \cdots \longleftarrow \bigoplus_j S(-j)^{\beta_{k,j}} \longleftarrow 0$$

if the complex is exact and all maps

$$\partial_k : \bigoplus_{a \in A} S(-a)^{\beta_{k,a}} \longrightarrow \bigoplus_{a \in A} S(-a)^{\beta_{k-1,a}}$$

are of degree zero. Such a resolution is minimal precisely when the entries in the ∂_k are in $\mathbf{m}R$, for $\mathbf{m}$ the homogeneous maximal ideal of R. For minimal resolutions $\mathcal{F}$ of M, the *Betti numbers* $\beta_{i,\mathbf{a}}(M)$ *of M* are invariants of the module. If $\mathbf{m} = x^{\mathbf{a}}$ is a monomial in R, we will frequently abuse our notation and denote $\beta_{i,\mathbf{a}}(M)$ by $\beta_{i,\mathbf{m}}(M)$.

Definition 2.1.13. Given a graded resolution $\mathcal{F}$ with graded syzygy modules $F_i = \oplus_j F_{i,j}$, the *linear strand of* $\mathcal{F}$ is

$$\mathcal{F}_{(1)} : \quad \cdots \longleftarrow F_{i-1,i-1} \longleftarrow F_{i,i} \longleftarrow F_{i+1,i+1} \longleftarrow \cdots .$$

The d*-strand of* $\mathcal{F}$ is

$$\mathcal{F}_{(d)}: \quad \cdots \longleftarrow F_{i-1,i+d-2} \longleftarrow F_{i,i+d-1} \longleftarrow F_{i+1,i+d} \longleftarrow \cdots.$$

An $\mathbb{N}$-graded R-module M has a *linear resolution in degree* d if its graded free resolution $\mathcal{F}$ is equal to the d-strand of $\mathcal{F}$.

Example 2.1.14 (Running Example). Let $R = \Bbbk[x,y,z,w]$, $I = (x^2, y^2, xz - yw)$ and $M = R/I$. Then a graded free resolution of M is given by

$$R/I \leftarrow R \xleftarrow{\varphi_1} R(-2)^3 \xleftarrow{\varphi_2} R(-4)^5 \xleftarrow{\varphi_3} R(-5)^4 \xleftarrow{\varphi_4} R(-6) \longleftarrow 0$$

where the maps are

$$\varphi_1 = \begin{pmatrix} x^2 & y^2 & xz - yw \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} -y^2 & 0 & xz - yw & yz & z^2 \\ x^2 & -xz + yw & 0 & -xw & -w^2 \\ 0 & y^2 & -x^2 & -xy & -xz - yw \end{pmatrix},$$

$$\varphi_3 = \begin{pmatrix} z & w & 0 & 0 \\ x & 0 & 0 & w \\ 0 & -y & -z & 0 \\ y & x & -w & -z \\ 0 & 0 & x & y \end{pmatrix},$$

and

$$\varphi_4 = \begin{pmatrix} -w \\ z \\ -y \\ x \end{pmatrix}.$$

In this case, the only nonzero Betti numbers are $\beta_{0,0}(R/I) = 1$, $\beta_{1,2}(R/I) = 3$, $\beta_{2,4}(R/I) = 5$, $\beta_{3,5}(R/I) = 4$, and $\beta_{4,6}(R/I) = 1$. These can be organized into a

Betti table (in the style of Macaulay 2) where the $\beta_{i,i+j}(R/I)$ is in the i^{th} column and j^{th} row.

-	0	1	2	3	4
total:	1	3	5	4	1
0:	1	·	·	·	·
1:	·	3	·	·	·
2:	·	·	5	4	1

Figure 2.1: Betti table of $I = (x^2, y^2, xz - yw)$.

These Betti tables will be returned to in Chapter 5, and feature prominently in proofs bounding the complexity of resolutions of R/I.

2.2 Algebraic Invariants of Modules

Studying the complexity of free resolutions is important for bounding the complexities of the modules themselves. While free resolutions of a module aren't unique, all minimal resolutions of a module are the same up to isomorphism. In particular, there are a few key algebraic invariants computed via resolutions which will be focused on here.

2.2.1 Projective Dimension, Regularity and Hilbert Functions

For the remainder, it will be assumed that we are working with A-graded R-modules M with $A = \mathbb{N}$ or $\mathbb{N}^n$ and $R = \Bbbk[x_1, ..., x_n]$. For fine gradings with $\mathbf{a} = (a_1, a_2, ..., a_n) \in \mathbb{N}^n$, we define $|\mathbf{a}|$ to be the sum of the entries $a_1 + a_2 + \cdots + a_n$.

Two invariants of M of particular interest are the projective dimension and regularity, which respectively measure the length and width of the resolution.

Definition 2.2.1. The *projective dimension of M*, $\mathrm{pd}(M)$, is

$$\mathrm{pd}(M) = \max\{i : \beta_{i,\mathbf{a}}(M) \neq 0\}.$$

The *(Castelnuovo-Mumford) regularity of M*, $\mathrm{reg}(M)$ is

$$\mathrm{reg}(M) = \max\{j - i : \beta_{i,\mathbf{a}}(M) \neq 0 \quad \text{with } |\mathbf{a}| = j\}.$$

Comparing free resolutions of R/I and I for ideal I as R-modules, we have

$$\mathrm{pd}(R/I) = \mathrm{pd}(I) + 1 \text{ and}$$

$$\mathrm{reg}(R/I) = \mathrm{reg}(I) - 1.$$

We also consider the following algebraic invariants: the Hilbert function, the Hilbert polynomial, and the Hilbert series of a module M.

Definition 2.2.2. Let $M = \oplus_{d \in \mathbb{N}} M_d$ be an $\mathbb{N}$-graded R-module. Then the *Hilbert function of M* is given by

$$\mathrm{Hilb}_M(d) := \dim(M_d),$$

the dimension over R of the d-th graded component of M. If for all $d \gg 0$, this agrees with a polynomial $P_M(x)$,

$$P_M(d) = \mathrm{Hilb}_M(d),$$

we say that $P_M(x)$ is the *Hilbert polynomial of M*. The *Hilbert series of M* is given by

$$\begin{aligned} H_M(t) &= \sum_{d \in \mathbb{N}} \mathrm{Hilb}_M(d) t^d \\ &= \frac{Q_M(t)}{(1-t)^n}, \end{aligned}$$

for some $Q_M(t) \in \mathbb{Z}[t]$ and $r \in \mathbb{N}$.

For a standardly graded polynomial ring $R = \Bbbk[x_1, ..., x_n]$ (i.e. $\deg(x_i) = 1$ for all i) and a homogeneous ideal $I \subseteq R$, we have that $P_M(x)$ exists for $M = R/I$. We extend this definition to multigraded R-modules.

Definition 2.2.3. Let $M = \oplus_{\mathbf{a} \in \mathbb{N}^n} M_{\mathbf{a}}$ be a multigraded module over a field $\Bbbk$. Then the *multigraded Hilbert function of M* is given by

$$\text{Hilb}_M(\mathbf{a}) := \dim_{\Bbbk} M_{\mathbf{a}},$$

or the dimension over the field of the $\mathbf{a}^{\text{th}}$ graded component of M.

The *multigraded Hilbert series of M* is given by

$$\begin{aligned} H_M(t_1, ..., t_n) &= \sum_{\mathbf{a} \in \mathbb{N}^k} \text{Hilb}_M(a_1, ..., a_n) t_1^{a_1} \cdots t_n^{a_n} \\ &= \frac{Q_M(t_1, ..., t_n)}{(1 - t_1) \cdots (1 - t_n)}, \end{aligned}$$

where $Q_M(t) \in \mathbb{Z}[t_1, ..., t_n]$.

For a survey of results on multigraded Hilbert series, see [34]. In this thesis, we will primarily consider the graded Hilbert polynomials of monomial ideals with the standard multigrading.

Example 2.2.4 (Running Example). Let $R = \Bbbk[x, y, z, w]$, $S = \Bbbk[z, w]$, $I = (x^2, y^2, xz - yw)$, and $M = R/I$. We read off the projective dimension and regularity directly from the Betti table computed in Example 2.1.14.

Similarly, from the calculations of the dimensions of $M = R/I$ performed in Example 2.1.6, we see that the Hilbert function and polynomial of R/I is given by

$$
\begin{aligned}
\mathrm{Hilb}_M(0) &= 1 \\
\mathrm{Hilb}_M(1) &= 4 \\
\mathrm{Hilb}_M(2) &= 7 \\
\mathrm{Hilb}_M(3) &= 8 \\
\vdots \quad & \quad \vdots \\
\mathrm{Hilb}_M(d) &= \mathrm{Hilb}_S(d-1) + \mathrm{Hilb}_S(d) + 1 \\
&= \binom{d}{d-1} + \binom{d+1}{d} + 1 = 2d+2.
\end{aligned}
$$

-	0	1	2	3	4
total:	1	3	5	4	1
0:	1	·	·	·	·
1:	·	3	·	·	·
2:	·	·	5	4	1

reg(R/I)=2
reg(I)=3

pd(R/I)=4
pd(I)=3

Figure 2.2: Projective Dimension and Regularity of R/I and I

So $P_M(d) = 2d + 2$. We can compute $H_M(t)$ either directly or from the Betti table via alternating sums of Betti numbers in the diagonals, obtaining

$$
\begin{aligned}
H_M(t) &= \frac{1 - 3t^2 + 5t^4 - 4t^5 + t^6}{(1-t)^4} \\
&= \frac{1 + 2t - 2t^3 + t^4}{(1-t)^2}.
\end{aligned}
$$

2.2.2 Monomial Ideals

Passing from I to its initial ideal $\text{in}_{\prec}(I)$ allows us to work with a monomial ideal, often providing a computational advantage. While much of the geometry of I is lost, various algebraic invariants and data from our module R/I can be recovered from these simpler ideals. We include here some basics of the theory of monomial ideals. For greater detail, see [20].

Definition 2.2.5. An ideal I is a monomial ideal if I is generated by monomials.

Monomial ideals are of interest due to their more accessible structure, in contrast to the often quite complex general case. Additionally, the relationship between a general ideal I and its initial ideals $\text{in}(I)$ provides a method of bounding various algebraic invariants of I.

Definition 2.2.6. Let $I \subseteq R = \Bbbk[x_1, ..., x_n]$ be an ideal and let $\prec$ be a term ordering on the set of monomials of R. Then for

$$f = \sum_{\mathbf{a} \in \mathbb{N}^n} c_{\mathbf{a}} x^{\mathbf{a}} \in I,$$

the *initial or leading term of f* or $\text{in}_{\prec}(f)$ is the monomial $c_{\mathbf{a}} x^{\mathbf{a}}$ in the support of f which is earliest under $\prec$. The ideal generated by the initial terms of I,

$$\text{in}_{\prec}(I) = \left\{\text{in}_{\prec}(f) : f \in I\right\}$$

is the *initial ideal of* I.

For an in-depth definition of term orderings and $in(I)$, see [25].

Passing to initial ideals $\text{in}_{\prec}(I)$ we can provide upper bounds on the entries in the Betti tables of general ideals I via the following theorem:

Theorem 2.2.7 (Upper-semicontinuity, Theorem 8.29 in [25]). Let $I \subseteq R = \Bbbk[x_1, ..., x_n]$ be a graded (respectively multigraded) ideal and $\text{in}_\prec(I)$ be its initial ideal. Then

$$\beta_{i,j}(R/I) \leq \beta_{i,j}(R/\text{in}_\prec(I))$$

for all $j \in A$ (and respectively $\beta_{i,\mathbf{a}}(R/I) \leq \beta_{i,\mathbf{a}}(R/\text{in}_\prec(I))$ for all $\mathbf{a} \in A$.)

Example 2.2.8 (Running Example). Let $R = \Bbbk[x, y, z, w]$, $I = (x^2, y^2, xz - yw)$, and let $\prec$ be the lex ordering. Then we have

$$\text{in}_\prec(I) = (x^2, y^2, xz, xyw)$$

with our Betti tables for the resolutions of I and $\text{in}_\prec(I)$ given by:

$\beta_{i,j}(I)$

-	0	1	2	3	4
total:	1	3	5	4	1
0:	1	·	·	·	·
1:	·	3	·	·	·
2:	·	·	5	4	1

$\beta_{i,j}(\text{in}_\prec(I))$

-	0	1	2	3	4
total:	1	3	5	4	1
0:	1	·	·	·	·
1:	·	3	1	·	·
2:	·	1	5	4	1

Figure 2.3: Comparison between Betti tables of I and $\text{in}_\prec(I)$.

More generally, passing from an ideal I to its initial ideal $\text{in}_\prec(I)$ preserves the Hilbert function, Hilbert polynomial and Hilbert series [30].

Another advantage to considering monomial ideals is the ability to predict potentially nonzero Betti numbers $\beta_{i,\mathbf{m}}(I)$ using the topology of the LCM-lattice via the following theorem:

Theorem 2.2.9 (LCM-lattice, Theorem 2.1 in [13]). Let I be a monomial ideal, L_I the lattice of least common multiples of generators of I, and $(\hat{0}, \mathbf{m})_{L_I}$ the open lower interval in the order complex of L_I. For $i \geq 1$ and $\mathbf{m} \in L_I$, we have

$$\beta_{i,\mathbf{m}}(S/I) = \dim \widetilde{H}_{i-2}\left(\left(\hat{0}, \mathbf{m}\right)_{L_I}; \Bbbk\right).$$

So if a monomial $\mathbf{m} \neq \operatorname{lcm}\{m_1, m_2, ..., m_r\}$ for generators $m_i \in I$ in monomial ideal I, $\beta_{i,\mathbf{m}}(I) = 0$.

General monomial ideals can be pushed to squarefree monomial ideals via polarization.

Definition 2.2.10 (Polarization). Let $I \subseteq R = \Bbbk[x_1, ..., x_n]$ be a monomial ideal with generators $m_1, ..., m_k$. The *polarization of a monomial* $m = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, denoted $\operatorname{pol}(m)$, is given by

$$\operatorname{pol}(m) := x_{1,1}x_{1,2}\cdots x_{1,a_1}x_{2,1}x_{2,2}\cdots x_{2,a_2}\cdots x_{n,1}x_{n,2}\cdots x_{n,a_n}.$$

The *polarization of a monomial ideal* I is

$$\operatorname{pol}(I) := (\operatorname{pol}(m_1), ..., \operatorname{pol}(m_k)).$$

Polarization of the resolution of a monomial ideal preserves syzygies, and hence, the polarization of a resolution $\operatorname{pol}(\mathcal{F})$ of a monomial ideal I gives the resolution of the polarization $\operatorname{pol}(I)$. Hence, polarization preserves Betti numbers, Hilbert functions, and numerous other invariants.[8] Additionally, it permits us to take a more topological approach, as detailed in the following section.

2.2.3 Stanley-Reisner Ideals and Edge/Hypergraph Ideals

In the case of square-free monomial ideals I_H, there are a number of ways of associating a simplicial complex to the ideal. The two complexes, Δ_{I_H} and Δ_H

are respectively the Stanley-Reisner complex and the facet complex. The first provides a fairly direct way of calculating the Betti numbers of I_H via Hochster's Formula, as in Theorem 3.2.9.

Definition 2.2.11. Let $I \subset R = \Bbbk[x_1, ..., x_n]$ be a square-free monomial ideal, also referred to as a *Stanley-Reisner ideal.* Then Δ_I, the Stanley-Reisner complex of I is the simplicial complex on vertex set $\{x_1, ..., x_n\}$ with faces

$$\{\sigma = \{x_{i_1}, ..., x_{i_r}\} \in \Delta : \mathbf{m} \nmid x_{i_1} \cdots x_{i_r} \forall \mathbf{m} \in I\}.$$

The second combinatorial correspondence referenced above, the *facet ideal* correspondence, puts the generators of our monomial ideal in a more direct correspondence with the facets of Δ.

Definition 2.2.12. Let $I_H = (m_1, ..., m_k) \subseteq R = \Bbbk[x_1, ..., x_n]$ be a square-free monomial ideal with minimal generators $\{m_1, ..., m_k\}$. Then the *facet complex of* I_H is the simplicial complex on vertex set $\{x_1, ..., x_n\}$ given by

$$\Delta_H = \{\{x_{i_0}, ..., x_{i_d}\} : x_{i_0} \cdots x_{i_d} | m_i \text{ for some } 1 \leq i \leq k\}.$$

So the Stanley-Reisner complex Δ_{I_H} has minimal non-faces corresponding to the monomial generators of I_H, while the maximal faces of the facet complex Δ_H directly correspond to these minimal monomials.

CHAPTER 3

BUILDING AND BOUNDING RESOLUTIONS

Broadly speaking, the techniques used here can be divided into two rough types - algebraic and topological. With the former, special emphasis will be placed on dividing up the ideal itself into smaller ideals then combining the two appropriately. In the latter, more often the Stanley-Reisner complex itself is partitioned into more manageable pieces and bounds are placed on the subcomplexes, followed by appropriate recombination into bounds on the entire complex.

3.1 Algebraic Techniques

While often for Stanley-Reisner ideals we make heavy use of the topology of their corresponding complexes, we often bound possible properties of $\mathbb{k}[\Delta]$ via algebra properties of I_Δ.

3.1.1 Taylor's Resolution

Let $I = (f_1, ..., f_k) =\subseteq R = \mathbb{k}[x_1, ..., x_N]$ be an ideal. For general (non-monomial ideals) we have that $\text{pd}(R/I) \leq N$. For monomial ideals, a different bound holds.

Theorem 3.1.1 (Taylor's Resolution [5])**.** Let $I = (m_1, ..., m_k)$ be a monomial ideal with $m_1, ..., m_k$ a minimal generating set of I. The Taylor resolution of I is $(T_\bullet(I), d_\bullet)$, defined by the following.

Set $T_j(I) = \bigwedge^{j+1} L$ for $j = 0, ..., k-1$, where L is the free R module with basis $\{e_1, ..., e_k\}$. Let $d_j : T_j(I) \longrightarrow T_{j-1}(I)$ for $j = 1, ..., k-1$, with

$$d_j(e_{i_0} \wedge \cdots \wedge e_{i_j}) = \sum_{r=0}^{j} (-1)^r \frac{\text{lcm}\{m_{i_0}, ..., m_{i_j}\}}{\text{lcm}\{m_{i_0}, ..., \widehat{m_{i_r}}, \cdots, m_{i_j}\}} e_{i_0} \wedge \cdots \wedge \widehat{e_{i_r}} \wedge \cdots \wedge e_{i_k}.$$

Then $(T_\bullet(I), d_\bullet)$ is a (possibly nonminimal) resolution of I.

Although this is not necessarily a minimal resolution, it provides a bound on projective dimension in terms of the number of generators.

Corollary 3.1.2. Let $I = (m_1, ..., m_k)$ be a monomial ideal. Then $\text{pd}(I) \leq k$.

This fails in the general case.

Example 3.1.3 (Running Example). The ideal $I = (x^2, y^2, xz - yw) \subseteq \Bbbk[x, y, z, w] = R$ has $\text{pd}(R/I) = 4$, but only 3 generators.

An immediate result of Theorem 3.1.1 is that the Betti numbers of monomial ideals are only potentially nonzero at the multidegrees of the least common multiples of generators of the monomial ideal. For the squarefree ideals that we consider, this implies all Betti numbers $\beta_{i,\mathbf{m}}(I_\Delta) = 0$ for any non-squarefree **m**.

As noted in [5], the resolution in Theorem 3.1.1 is often far from minimal and the problem of finding explicit minimal resolutions of monomial ideals is an active research problem.

3.1.2 Rees Algebras and Degree Restrictions

One common technique used in investigating powers I^n of an ideal I involves passing to the Rees algebra of I. The Rees algebra $\mathcal{R}(I)$ of an ideal I is an object which captures the ideal I and all of its powers.

Definition 3.1.4. Let $I = (f_0, f_1, ..., f_k) \subseteq R = \Bbbk[x_1, ..., x_N]$. The *Rees algebra* $\mathcal{R}(I)$ *of* I is

$$\mathcal{R}(I) = R \oplus It \oplus I^2t^2 \oplus I^3t^3 \oplus \cdots \oplus I^nt^n \oplus \cdots$$

This is occasionally denoted $R[It]$.

In general, we will use a presentation of $\mathcal{R}(I)$ as a quotient module of the ring $S = R[w_0, w_1, ..., w_k] = \Bbbk[x_1, ..., x_N, w_0, w_1, ..., w_k]$.

Proposition 3.1.5 (Proposition 10.2.11 in [36])**.** Let $I = (f_1, ..., f_k) \subseteq R = \Bbbk[x_1, ..., x_N]$ and let $\mathcal{R}(I)$ be its Rees algebra. Then $\mathcal{R}(I) = R[w_1, ..., w_k]/L = \Bbbk[x_1, ..., x_N, w_0, w_1, ..., w_k]/L$, with presentation ideal

$$L = (f_i - w_it : 1 \leq i \leq k)S[t] \cap S.$$

If $S = \Bbbk[x_1, ..., x_N, w_1, ..., w_k]$, and $\mathcal{R}(I) = S/L$, then L is the *Rees ideal of* I.

Taking a resolution (with an appropriately chosen bigrading) of L gives resolutions of all powers of L, and can be used to bound or explicitly compute Betti numbers $\beta_{i,j}(I^n)$ for all n. We make use of this technique in Chapter 7 to show that the shape of the Betti tables of all higher powers of equigenerated ideals stabilizes.

The Rees ideal of I can be computed via Proposition 3.1.5, but an explicit characterization of the generating set of L for edge ideals is due to Villareal.

Definition 3.1.6. Let G be a simple undirected graph and $p = \{v_0, ..., v_n\}$ an even closed walk in G, with edges $f_i = v_{i-1}v_i$ in G corresponding to variables w_i in the Rees algebra of the edge ideal I_G. Then

$$w_p = w_1w_3 \cdots w_{n-1} - w_2w_4 \cdots w_n$$

we call the *binomial coming from p.*

Theorem 3.1.7 (Rees Ideals of Edge Ideals, Theorem 3.1 and Proposition 3.1 in [38])**.** Let G be a graph with let $F = \{f_1, ..., f_q\}$ be the set of edge generators of I_G. Let $\mathcal{R}(I) = S/L$ be the presentation of the Rees algebra with respect to F. Then

$$L = SL_1 + S \cdot \left(\cup_{s=2}^{\infty} P_s\right),$$

where L_1 is the degree 1 (in the w_i-variables) homogeneous component of L, and where $P_s = (w_p = w_1 w_3 \cdots w_{s-1} - w_2 w_4 \cdots w_s :$ p is an even closed path of length s.)

3.2 Topological Techniques and Additional Definitions

Definition 3.2.1 (Links and Stars)**.** Let $\sigma \in \Delta$ be a face of Δ. The *link of* σ *in* Δ is

$$\text{link}_{\Delta}(\sigma) := (\tau \in \Delta : \tau \cup \sigma \in \Delta, \tau \cap \sigma = \emptyset).$$

The *star of* σ *in* Δ is

$$\text{star}_{\Delta}(\sigma) := (\tau \in \Delta : \tau \cap \sigma \neq \emptyset).$$

Example 3.2.2. Let Δ be the simplicial complex in Figure 3.1, with $v \in \Delta$ the central vertex. Then the link$_{\Delta}(v)$ and star$_{\Delta}(v)$ are given in Figure 3.1. Note that star$_{\Delta}(v)$ is not a simplicial complex.

We also introduce the k-skeleton of a simplicial complex.

Definition 3.2.3 (k-skeleta)**.** Let Δ be a simplicial complex. Then the *k-skeleton of* Δ is

$$\Delta_k := \{\sigma \in \Delta : |\sigma| \leq k + 1\}.$$

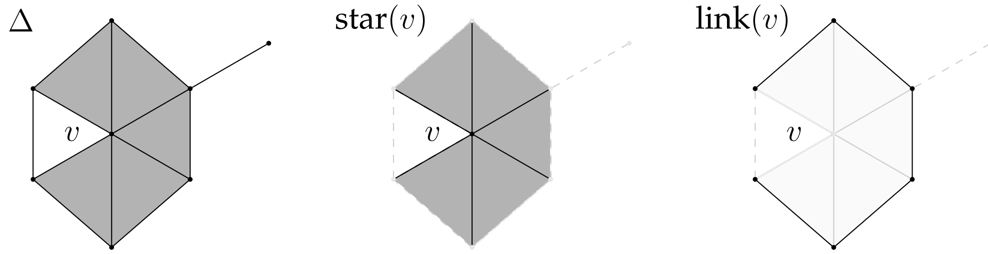


Figure 3.1: Example 3.2.2: Star and Link of a vertex $v \in \Delta$.

Frequently, we consider the 1-skeleton of Δ in the case of edge ideals.

Definition 3.2.4. We say that a *minimal vertex cover* of a simplicial complex Δ is a set of vertices $\mathcal{C} = \{v_{i_1}, ..., v_{i_k}\} \in \Delta$ such that every face $\sigma \in \Delta$ has at least one $v \in \sigma$ from $\mathcal{C}$, and no subset $\mathcal{C}' \subseteq \mathcal{C}$ has this property.

Most commonly, we will use minimal vertex covers of graphs G, rather than general simplicial complexes Δ. Computing the minimal vertex covers of G gives us the primary decomposition of squarefree ideal I_G.

Proposition 3.2.5 (Primary Decomposition of Facet Ideals, Proposition 1 in [7])**.** Let Δ be a simplicial complex on n vertices. Let $I_{\mathcal{F}(\Delta)}$ be its facet ideal in polynomial ring $\Bbbk[x_1, ..., x_n]$ over a field $\Bbbk$. Then an ideal $p = (x_{i_1}, ..., x_{i_s})$ of R is a minimal prime of I if and only if $\{x_{i_1}, ..., x_{i_s}\}$ is a minimal vertex cover for Δ.

Final notations we introduce in this section are the f-vector and h-vector, which will also play large roles in describing algebraic invariants of I_Δ and $I_{\mathcal{F}}$, the Stanley-Reisner ideal of Δ and the facet ideal of Δ, respectively.

Definition 3.2.6 (f-vector and h-vector of Δ)**.** Given a simplicial complex Δ, let f_i denote the number of faces of dimension i. The *f-vector of* Δ is $f = (f_{-1}, f_0, f_1, ..., f_d)$, where by convention $f_{-1} = 1$. The *h-vector of* Δ,

$h = (h_0, h_1, ..., h_d)$, is given by

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}.$$

Theorem 3.2.7 (Numerator of Hilbert Series for Stanley-Reisner Rings, Theorem 1.13 in [25])**.** Let Δ a simplicial complex on n vertices, and $I_\Delta \subseteq \mathbb{k}[x_1, ..., x_n]$ its Stanley-Reisner ideal. The numerator of the Hilbert series of the Stanley-Reisner ring S/I_Δ is a polynomial $Q_{S/I_\Delta}(\mathbf{x})$, given by

$$Q_{S/I_\Delta}(\mathbf{x}) = \sum_{\sigma \in \boldsymbol{\Delta}} \left(\prod_{\mathbf{i} \in \sigma} \mathbf{x_i} \cdot \prod_{\mathbf{j} \notin \sigma} (\mathbf{1} - \mathbf{x_j}) \right).$$

The connection between this formula and the f-vector and h-vector of Δ is given in the following corollary.

Corollary 3.2.8 (Hilbert Series of S/I_Δ, Corollary 1.15 in [25])**.** Let $f = (f_{-1}, f_0, f_1, ..., f_{d-1})$ be the f-vector of Δ, $\dim(\Delta) = d-1$, and let I_Δ be its Stanley-Reisner ideal. Then

$$\begin{aligned} H_{S/I_\Delta}(t) &= \frac{1}{(1-t)^n} \sum_{i=0}^{d} f_{i-1} t^i (1-t)^{n-i} \\ &= \frac{h_0 + h_1 t + h_2 t^2 + \cdots + h_d t^d}{(1-t)^d}. \end{aligned}$$

We use this characterization of the Hilbert series of S/I_Δ in Section 9.3.

3.2.1 Hochster's Formula

A key result used here in producing the Betti numbers of squarefree monomial ideals is Hochster's formula, phrased here in terms of the ranks of the homologies of induced subcomplexes of the Stanley-Reisner complex of I_Δ.

Theorem 3.2.9 (Hochster's Formula [19])**.** Let I_Δ be a squarefree monomial ideal in variables $X = \{x_1, ..., x_n\}$ and let Δ be its Stanley-Reisner complex. Then if $\mathbf{m}$ is a squarefree monomial with support $W = \{x_{i_1}, ..., x_{i_j}\} \subseteq X$ with $\deg(m) = j$, we have

$$\beta_{i,\mathbf{m}}(k[\Delta]) = \dim \widetilde{H}_{j-i-1}(\Delta|_W, k),$$

where $\Delta|_W$ is the induced subcomplex of Δ on vertices in W.

While this is often phrased in terms of the homologies of the links of induced subcomplexes of the Alexander dual of Δ, the formulation in Theorem 3.2.9 is convenient for our purposes. We now introduce numerous tools for decomposing and examining complexes Δ in terms of their subcomplexes.

3.2.2 Mayer-Vietoris Sequences

In general, given a topological space X with two subspaces A and B such that $X = A \cup B$, the Mayer-Vietoris sequence relates the homologies of A, B, and $A \cap B$ to the homology of X. The modified version here is used for decompositions of simplicial complexes into vertex induced subsets.

Theorem 3.2.10 (Mayer-Vietoris Theorem, Section 2.2 in [17])**.** Let Δ be a d-dim'l simplicial complex on vertex set X. Let $X = A \cup B$ be two subsets of the vertices of X such that $\Delta|_A \cup \Delta|_B = \Delta$, i.e. the induced subcomplexes on vertex sets A and B include all faces of Δ. Then we have the following long exact sequence of homology groups:

$$\begin{gathered}
0 \to H_d(\Delta|_A) \oplus H_d(\Delta|_B) \to H_d(\Delta) \xrightarrow{\partial} H_{d-1}(\Delta|_{A\cap B}) \to \cdots \\
\cdots \to H_i(\Delta|_{A\cap B}) \to H_i(\Delta|_A) \oplus H_i(\Delta|_B) \to H_i(\Delta) \xrightarrow{\partial} H_{i-1}(\Delta|_{A\cap B}) \to \cdots \\
\cdots \to H_1(\Delta) \xrightarrow{\partial} H_0(\Delta|_{A\cap B}) \to H_0(\Delta|_A) \oplus H_0(\Delta|_B) \to H_0(\Delta) \to 0
\end{gathered}$$

Example 3.2.11. Let Δ be the simplicial complex in Figure 3.2 on vertex set $X = \{a, b, c, d, e\}$. Then the subsets $A = \{a, b, c, d\}$ and $B = \{b, c, d, e\}$ cover X, but do not satisfy that $\Delta|_A \cup \Delta|_B = \Delta$. The edge $\{a, e\}$ is not included in $\Delta|_A \cup \Delta|_B$, and Theorem 3.2.10 fails to hold.

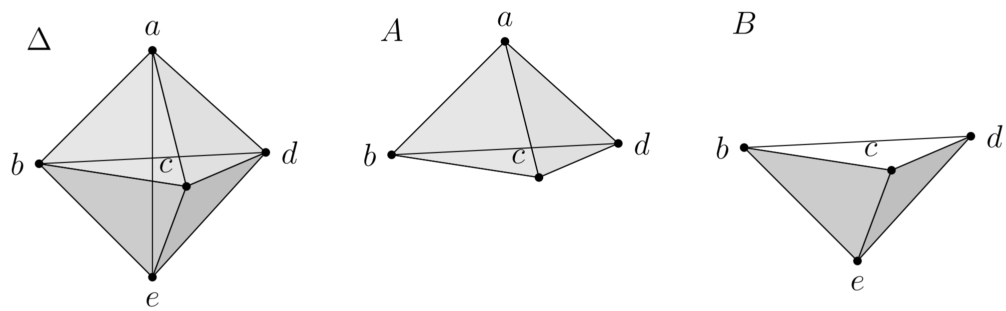


Figure 3.2: Failed simplicial Mayer-Vietoris decomposition.

This will be our primary computational tool in Chapter 4.

3.2.3 Discrete Morse Theory

Discrete Morse Theory, developed by Robin Forman for a broad class of cellular complexes in [10] and [11], provides a combinatorial framework in which to reduce a given (simplicial, cellular, regular CW, etc.) complex into a simpler homotopically equivalent complex. We will not make use of the full power of the theory here. The class of complexes we will apply this to in Chapter 9 are all simplicial, so a more abbreviated version of the theory will do. We use the language of Chari, as in [31].

Let Δ be a finite simplicial complex with faces $F_\Delta = \{\sigma \in \Delta\}$ and $D(\Delta)$ the directed graph of the face poset of Δ, or *face digraph of* Δ. This is the digraph on vertex set F_Δ with a directed edge (τ, σ) between two faces σ and τ if and only if $\sigma \subseteq \tau$. The set of all such arcs we will denote A_Δ. In $D(\Delta) = \{F_\Delta, A_\Delta\}$, the vertices τ and σ will be referred to as *endpoints of the arc* (τ, σ).

Given any subset $M \subseteq A_\Delta$, we define $M^{op} := \big\{(\sigma, \tau) : (\tau, \sigma) \in M\big\}$ and form the new digraph

$$D_M(\Delta) := \{F_\Delta, A_\Delta \backslash M \cup M^{op}\}.$$

Definition 3.2.12. A *Morse matching* on $D(\Delta)$ is a collection of arcs $M \subseteq A_\Delta$ which satisfies the following two conditions:

(M1) each $\sigma \in F_\Delta$ is an endpoint of at most one arc in M (**M a matching**),

(M2) the digraph $D_M(\Delta)$ contains no directed cycle (**M acyclic**).

Any faces $\sigma \in F_\Delta$, $\dim(\sigma) = d$, which are not the endpoint of any arc in M are called *M-critical d-cells*.

The application of constructing such a matching is seen in the following theorem:

Theorem 3.2.13 (Theorem 2.5 in [11]). Suppose K is a simplicial complex with discrete Morse matching M. Then K is homotopy equivalent to a CW-complex with exactly one cell of dimension d for each M-critical simplex of dimension d.

We use this in Section 9.2 to provide a self-contained proof of the Nerve theorem for neighborhood complexes of graphs.

3.2.4 Alexander Duality

We define first combinatorially duality, then extend this construction to a dual on our squarefree ideals which interchanges projective dimension and regularity.

Definition 3.2.14. Let Δ be a simplicial complex. Its *Alexander dual*

$$\Delta^* = \{\bar{\tau} : \tau \notin \Delta\}$$

consists of the complements of nonfaces of Δ.

In the following proposition, let $\mathbf{x}^\sigma$ denote the monomial $x_{i_1} \cdots x_{i_{d+1}}$ corresponding to $\sigma = \{v_{i_1}, ..., v_{i_{d+1}}\}$, a face of dimension d. Let $\mathfrak{m}^\sigma = (x_{i_1}, ..., x_{i_{d+1}})$ be the ideal generated by all variables dividing x^σ.

Proposition 3.2.15. Let I be a squarefree ideal. Then the *squarefree Alexander dual* of $I = (\mathbf{x}^{\sigma_1}, ..., \mathbf{x}^{\sigma_1})$ is

$$I^\vee = \mathfrak{m}^{\sigma_1} \cap \cdots \cap \mathfrak{m}^{\sigma_{d+1}}.$$

Alexander duality interchanges projective dimension and regularity for Stanley-Reisner ideals.

Proposition 3.2.16 (Theorem 5.59, [25])**.** Let $I \subseteq R$ be a squarefree ideal. Then the regularity of I equals the projective dimension of $R/I^{\vee}$.

3.3 Edge Ideals

Let G be a simple graph (e.g. undirected with no loops or multiple edges) on n vertices, and e edges, denoted $G = (V, E)$ with V the vertex set of G and E the edge set. Let $\Bbbk$ be a field of characteristic zero and $R = \Bbbk[x_1, .., x_n]$ the polynomial ring over $\Bbbk$ with a generator for each vertex of G.

Definition 3.3.1. Let G, R as above. Then the *edge ideal of* G, denoted I_G is the squarefree monomial ideal given by

$$I_G = (x_i x_j : \{i, j\} \in E).$$

Ideals of this form have been of great interest recently, with excellent surveys in [15] and [16]. After their introduction by Villareal [37], they have been studied extensively, with the goal of building a dictionary between graph properties of G and algebraic properties of I_G.

In the case of edge ideals I_G, their Stanley-Reisner complexes, Betti numbers, and shape of their Betti tables are determined by properties of the graph G.

3.3.1 Flag Complexes and Clique Closures

Definition 3.3.2. The *clique complex* $\widehat{G}$ *of a graph* G is the simplicial complex on the vertex set of G whose facets are the maximal cliques, or maximal complete subgraphs, of G. The *clique closure of a simplicial complex* Δ denoted $\widehat{\Delta}$, is the complex obtained by closing the complex under the operation of iteratively adding a face σ to Δ whenever $\partial\sigma \subseteq \Delta$ and $|\sigma| > 2$.

Example 3.3.3. The simplicial complex Δ in Figure 3.3 has facets

$$\mathcal{F}_{\Delta} = \big\{\{a,b\},\{a,h\},\{b,h,e\},\{b,c\},\{b,e\},\{b,g\},\{c,d\},\{d,e\},\{e,g\},\{g,h\},\{f\}\big\}$$

and has clique closure $\widehat{\Delta}$. The facets of $\widehat{\Delta}$ are

$$\mathcal{F}_{\widehat{\Delta}} = \big\{\{a,b,h\},\{b,e,g,h\},\{b,c\},\{c,d\},\{d,e\},\{f\}\big\}.$$

Note that the condition that $|\sigma| > 2$ in Definition 3.3.2 does not require the

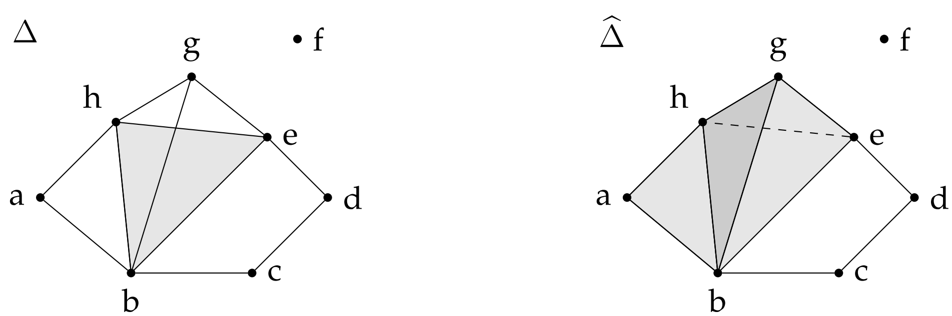


Figure 3.3: Complexes Δ and $\widehat{\Delta}$

addition of such faces as $\sigma = \{b,f\}$ to $\widehat{\Delta}$, although $\{b\}$ and $\{f\}$ are both in Δ.

Remark 3.3.4. Complexes with $\Delta = \widehat{\Delta}$ satisfy the properties in Definition 3.3.2 and will also be referred to as *clique complexes* or *flag complexes*.

We will use the following well-known results in the calculation of our Betti numbers and in the characterization of our graphs and simplicial complexes:

Proposition 3.3.5. Let G be a graph, I_G its edge ideal, and Δ_G its Stanley-Reisner complex. Then the following hold:

1. Δ_G is clique closed, i.e. $\Delta_G = \widehat{\Delta_G}$, and

2. Its 1-skeleton is the complement graph of G, $(\Delta_G)_1 = G^c$.

Proof. As I_G is generated in degree 2, all minimal nonfaces of the Stanley-Reisner complex are edges. Assume $T = \{\tau : \tau \in \partial\sigma\}$ for some face $\sigma \in \Delta_G$ with $|\sigma| > 2$ be in Δ. As all minimal nonfaces are of dimension 2 and $\partial\sigma \subseteq \Delta$, $\tau \in \Delta$ as τ cannot be a minimal nonface. So (1) holds.

Every minimal nonface of Δ_G is an edge in G, so the 1-skeleton of Δ_G is precisely the edges not in G. Hence, (2) holds. □

A related proposition gives that all clique complexes have Stanley-Reisner ideals arising as edge ideals.

Proposition 3.3.6. Given any clique complex $\Delta = \widehat{\Delta}$, its Stanley-Reisner ideal I_Δ is squarefree and generated in degree 2. Hence, there exists a graph G such that $I_G = I_\Delta$.

Proof. Given a clique complex Δ, assume that $\sigma = \{x_{i_1}, x_{i_2}, ..., x_{i_k}\}$ is a minimal nonface. If $k > 2$, then by $\Delta = \widehat{\Delta}$, we have all subsets

$$\sigma_j = \{x_{i_1}, x_{i_2}, ..., \widehat{x_{i_j}}, ..., x_{i_k}\}$$

in Δ, which contradicts σ a minimal non-face. As all of our minimal nonfaces are assumed to be of size $k \geq 2$ (as $\{v : v \in V\} \in \Delta$ is assumed in our definition of simplicial complex,) we have that all minimal non-faces $\sigma = \{x_{i_1}, x_{i_2}\}$, and hence, I_Δ is generated in degree 2. □

As the Stanley Reisner complex of the ideal I_G is exactly the clique complex of the complement graph G^c, properties of the complement graph feature heavily in determinations of the resolutions of I_G. We denote the Stanley Reisner complex of I_G as Δ_G or $\widehat{G^c}$ throughout.

Combining this characterization with Theorem 3.2.9, we have Hochster's formula for the Betti numbers of edge ideals.

Proposition 3.3.7. Let G be a simple graph on vertex set $[n] = \{1, 2, ..., n\}$ with edge set E, and let $I_G = (x_i x_j : \{i, j\} \in E) \subseteq R = k[x_1, ..., x_n]$ be the edge ideal of G. Then the Stanley-Reisner complex of I_G, denoted $\Delta(I_G)$, is given by

$$\Delta(I_G) = \widehat{G^c},$$

the clique closure of the complement graph of G in $[n]$. So

$$\beta_{i,\mathbf{m}}(I_G) = \dim \widetilde{H}_{j-i-1}(\widehat{G^c}|_{\mathbf{m}}, \Bbbk).$$

These will form the primary basis of our Betti number calculations which we return to in subsequent sections.

CHAPTER 4

REGULARITY AND PROJECTIVE DIMENSION BOUNDS

4.1 Edge Independence Number

Definition 4.1.1. We say that G has induced matching number k, or

$$\text{indMatch(G)} = k$$

if the largest subset of edges that can be chosen to be mutually disjoint in the induced subgraph of G restricted to those vertices is of size k. We say that G has matching number k, or

$$M(G) = k$$

if the largest mutually disjoint set of edges is of size k.

Example 4.1.2. Considering the cycle graphs of lengths 4, 5, and 6 as seen in Figure 4.1, denoted C_4, C_5 and C_6 respectively, we see that

$$\text{indMatch}(C_4) = 1, \text{indMatch}(C_5) = 1, \text{indMatch}(C_6) = 2$$

and

$$M(C_4) = 2, M(C_5) = 2, \text{ and } M(C_6) = 3.$$

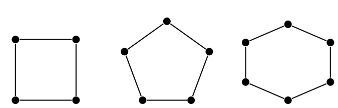


Figure 4.1: Cycle Graphs

Figure 4.2: Induced Matchings on Cycle Graphs

For these graphs, the regularity of I_G is respectively $\text{reg}(I_{C_4}) = 2$, $\text{reg}(I_{C_5}) = 3$, and $\text{reg}(I_{C_6}) = 3$.

4.2 Lower Bounds on Regularity of Edge Ideals

We use Proposition 3.3.5 to reformulate statements about the induced matching number indMatch(G) of G in terms of properties of Δ_G.

Recall first the definition of the boundary of the cross polytope.

Definition 4.2.1. The boundary $\partial\beta_{k+1}$ of *k-dimensional cross polytope* β_{k+1} is given by

$$\partial\beta_{k+1} = \overbrace{S_0 * S_0 * \cdots * S_0}^{k+1} \subseteq \Delta_G,$$

for S_0 a set consisting of two points. We may represent this as a simplicial complex on vertex set $V_{k+1} = \{x_1, y_1, x_2, y_2, ..., x_{k+1}, y_{k+1}\}$ where the faces of $\partial\beta_{k+1}$ are all subsets $\sigma \in 2^{V_{k+1}}$ such that for each i, at most one of x_i or y_i is in σ.

Proposition 4.2.2. Let G be a simple graph with edge ideal I_G and Stanley-Reisner complex Δ_G. The following are equivalent:

1. indMatch(G) $= k$
2. Δ_G has a set of vertices V_k such that $\Delta_G|_{V_k} = \partial\beta_{k+1}$ and no $\partial\beta_r$ for $r > k+1$ is an induced subcomplex of Δ_G.

Proof. We prove a slightly stronger statement. If E is any set of edges of size r in G with the induced graph G on those edges completely disconnected, we have that Δ_G contains β_{r+1}, and vice versa. This is equivalent to proving that the Stanley-Reisner complex of a graph consisting of r disjoint edges is the boundary of the r-dimensional cross polytope, as all properties of these complexes rely only on combinatorial data of induced subgraphs and subcomplexes.

Without loss of generality, let G be the graph consisting of r disjoint edges, with edge set $E = \{\{x_1, y_1\}, \{x_2, y_2\}, ..., \{x_r, y_r\}\}$. By definition, each edge in G is a minimal nonface of Δ_G, and all faces containing at most one vertex in each edge-pair must be in Δ_G. So the facets $\mathcal{F}$ of Δ_G must be of the form

$$\mathcal{F} = \{\sigma = \{w_1, w_2, ..., w_r\} : w_i = x_i \text{ or } w_i = y_i\}.$$

This is precisely the boundary of the r-dimensional cross polytope. $\square$

Example 4.2.3. For the graph G consisting of 3 disjoint edges in Figure 4.3, we see $\Delta_G = \partial\beta_3 \cong S^2$.

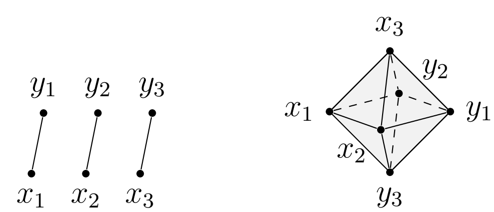


Figure 4.3: indMatch(G)=3, $\Delta_G \cong S^2$

Versions of Proposition 4.2.2 can be found in [4], [22], [28], [40] and [41]. The fact that the indMatch(G) forms a lower bound on the regularity of I_G is immediate from Proposition 4.2.2 and Hochster's formula.

Proposition 4.2.4 (Theorem 2.18 in [41])**.** Let G be a graph with edge ideal I_G. Then

$$\text{reg}(I_G) \geq \text{indMatch(G)} + 1.$$

If G is a tree, equality holds.

This inequality was proved in the special case when G is a tree by Zheng in [41], and equality was proved for all chordal graphs G by Tai Ha and Van Tuyl.

Theorem 4.2.5. [Corollary 6.9 in [16]] Let G be a chordal graph. Then

$$\text{reg} I_G = \text{indMatch(G)} + 1.$$

4.3 Rough Upper Bounds on Regularity

The equality in Theorem 4.2.5 fails to hold for nonchordal graphs in even the simplest of cases. For example, for the 5-cycle C_5, we have IndMatch$(C_5) = 1$ but reg$(I_{C_5}) = 3$. Much work of late has been focused on producing upper bounds on the regularity of reg(I_G) in terms of combinatorial data on G. One known rough upper bound is the matching number of G, $M(G)$.

Proposition 4.3.1 (Theorem 6.7 in [16])**.** Let G be a finite simple graph. Then

$$\text{reg}(I_G) \leq M(G) + 1$$

where $M(G)$ is the matching number of G, i.e. the largest size of a maximal matching in G.

Some sharper bounds are known in other classes of simple graphs. A *claw* in a graph G is an induced subgraph on four vertices isomorphic to the star graph on four vertices, . A graph with no such induced subgraph is said to be *claw-free*. In the case of claw-free graphs G with indMatch(G) $= 1$, the regularity is given precisely by the following theorem:

Theorem 4.3.2 (Theorem 1.2 in [28])**.** Let G be claw free such that G^c has no induced 4-cycle. Then:

(1) I_G^2 has a linear resolution.

(2) If G^c is not chordal, then $\text{reg}(I_G) = 3$.

Another such bound on regularity of I_G can be found in [40]. A *complement chordal graph* G_i is a graph such that G_i^c is chordal. The smallest r such that G is covered by complement chordal graphs G_i,

$$G = \cup_{i=1}^{r} G_i$$

is the co-chordal number of G, $\text{cochord}(G)$.

Theorem 4.3.3 (Theorem 13 in [40])**.** For any graph G, we have

$$\text{reg} I_G \leq \text{cochord}(G) + 1.$$

Bounding the regularity of the subclass of graphs with indMatch(G) $= 1$, and classifying the regularity and types of Betti diagrams of such edge ideals I_G will provide us with tools for bounding regularity of general graphs G with indMatch(G) $= k$. We note here three equivalent characterizations of such graphs, obtained by combining Proposition 4.2.2 with Proposition 3.3.7 to characterize graphs G with indMatch(G) $= 1$.

Corollary 4.3.4. Let G be a simple graph, G^c its complement graph and I_G its edge ideal. The following are equivalent:

1. indMatch(G) $= 1$,
2. G^c has no induced 4-cycles, and
3. $\beta_{2,4}(I_G) = 0$.

Proof. We have that $(1) \Leftrightarrow (2)$ from Proposition 4.2.2 with $k = 1$. Using Proposition 3.3.7 for graded modules, we have that

$$\beta_{2,4}(I_G) = \sum_{\substack{W \subseteq V \\ |W|=4}} \dim \widetilde{H}_1(\widehat{G^c}|_W, \Bbbk).$$

This Betti number is precisely nonzero when the $\widehat{G^c}|_W$ has no cycles of length 4 in the complement graph G if we restrict to any set of vertices of G of size 4. By Proposition 4.2.2, this is true precisely when there are no pairs of induced disjoint edges in our original graph, as a 4-cycle is the one dimensional cross polytope. $\square$

Figure 4.4: Graph G and Δ_G

We conjecture here a bound on the regularity of edge ideals with indMatch(G) $= 1$, and a more general conjecture for a bound on the regularity of edge ideals with indMatch(G) $= k$.

Conjecture 4.3.5. Let G be a simple graph and I_G its edge ideal. If indMatch(G) $= 1$, then

$$\text{reg}(I_G) \leq 5.$$

If indMatch(G) $= k$, then

$$\text{reg}(I_G) \leq 4k + 1.$$

Example 5.1.2 is a convex simplicial 4-polytope, and hence, its Stanley-Reisner complex is shellable and Gorenstein. So regularity bounds in these classes cannot be sharpened below reg$(I_G) \leq 5$. There also exist Gorenstein graphs G with indMatch(G) $= 1$, reg$(I_G) = 4$ and arbitrarily high projective dimension, which we construct in Section 6.4.

Conjecture 4.3.5 is still open. However, Example 5.1.2 has IndMatch$G_k = k$ and reg$(I_{G_k}) = 4k + 1$, which implies that no tighter bound is possible.

CHAPTER 5

BETTI NUMBERS AND JUMP SEQUENCES

Given an edge ideal of simple graph G, we show that if the first nonlinear strand in the resolution of I_G is zero until homological stage a_1, then the next nonlinear strand in the resolution is zero until homological stage $2a_1$. Additionally, we define a sequence, called a *jump sequence*, characterizing the highest degrees of the free resolution of the edge ideal of G via the lower edge of the Betti diagrams of I_G.

These sequences strongly characterize topological properties of the underlying Stanley-Reisner complexes of edge ideals, and provide general conditions on construction of clique complexes on a fixed set of vertices. We also provide an algorithm for obtaining a large class of realizable jump sequences and classes of Gorenstein edge ideals achieving high regularity.

Question 5.0.6. [Open] For graphs G with indMatch(G) $= 1$ is there a bound on the regularity of I_G? This is equivalent to bounding regularity for graphs G such that G^c is induced 4-cycle free or $\beta_{2,4}(I_G) = 0$ as in Corollary 4.3.4.

As discussed in Section 4.3, partial answers exist. Examples of graphs which have indMatch(G) $= 1$ and regularity of I_G up to 5 are known, and we provide new general classes of graphs with indMatch(G) $= 1$ and regularity $I_G = 4$. We also provide an example of graphs with indMatch(G) $= k$ and regularity as high as indMatch(G) $= 4k + 1$.

We introduce the following pair of invariants of I_G.

Definition 5.0.7. Let G a graph, I_G its edge ideal, and $\beta_{i,j} = \beta_{i,j}(I_G)$ as above. If

I_G has $\text{reg}(I_G) = k+1$, then I_G has a *jump sequence Jump*(I_G) *of length k-1* of the form

$$\text{Jump}(I_G) = [k; a_1, ..., a_{k-1}],$$

where $a_r = \min\{i : \beta_{i,i+r+1} \neq 0\} - 1$. If I_G has a linear resolution, we say $\text{Jump}(I_G) = [1; \emptyset]$.

Definition 5.0.8. Let G be a graph, I_G its edge ideal, and $\text{Jump}(I_G)$ its jump sequence. Then the *relative jump sequence relJump*(I_G) *of* G is

$$\text{relJump}(I_G) = [k; r_1, r_2, ..., r_{k-1}],$$

where $r_1 = a_1$ and $r_i = a_i - a_{i-1}$ for all $i = 2, .., k-1$.

-	0	1	2	3	a_1+1	$\cdots$	a_2+1	$\cdots$	$a_{k-1}-1$	a_{k-1}	$a_{k-1}+1$
total:	1	β_1	β_2	$\cdots$	β_{a_1+1}	$\cdots$	β_{a_2+1}	$\cdots$	$\beta_{a_{k-1}-1}$	$\beta_{a_{k-1}}$	$\cdots$
0:	1	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	
1:	$\cdot$	$\beta_{1,2}$	$\beta_{2,3}$	$*$	$*$	$*$	$*$	$*$			
2:			a_1		β_{a_1+1,a_1+3}	$*$	$*$	$*$	$*$		
3:				a_2			β_{a_2+1,a_2+4}	$*$	$*$	$*$	$*$
$\vdots$									$*$	$*$	
k:						a_{k-1}					$\beta_{a_{k-1}+1,s}$

Figure 5.1: General Betti table of Edge Ideal I_G

This idea of a *staircase* walking down the highest degree Betti numbers leads to the above definition. If we have a Betti table of the form in Figure 5.1, where $s = a_{k-1} + k + 1$, with $\beta_{a_i+1,a_i+i+1} \neq 0$ and all Betti numbers below the line are zero, then G (or I_G) has the jump sequence $\text{Jump}(I_G) = [k; a_1, a_2, ..., a_{k-1}]$.

Note that $a_1 + 1$ marks the homological degree of the resolution where the first nonlinear syzygies occur, having degree 1 above linear syzygies. Similarly, a_2+1 reads off the homological degree where the first syzygies two degrees above linear occur. In general, $a_i + 1$ is the homological degree where the first syzygies occur which are degree i above linear syzygies.

We provide a few examples to motivate Jump(I_G).

Example 5.0.9. Let G be the graph of the anticycle of length n, e.g. $G^c = C_n$. The Betti diagram is of the form:

$$
\begin{array}{ccccccccc}
- & 0 & 1 & 2 & 3 & \cdots & n-4 & n-3 & n-2 \\
\text{total:} & 1 & \beta_1 & \beta_2 & \beta_3 & \cdots & \beta_{n-4} & \beta_{n-3} & \beta_{n-2} \\
0: & 1 & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
1: & \cdot & * & * & * & \cdots & * & * & \cdot \\
2: & & \multicolumn{6}{c}{a_1 = n-3} & 1
\end{array}
$$

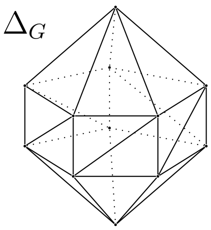
Figure 5.2: Betti Tables of the Edge Ideal of the Anticycle I_{C_n}

So all singleton sequences are possible, with Jump$(I_G) = [2; n]$ for the edge ideal of the complement graph of the $n + 3$ cycle.

Example 5.0.10. Let G be the graph with edge ideal

$$
\begin{aligned}
I_G = (&x_1x_3, x_1x_4, x_2x_4, x_2x_5, x_3x_5, x_2x_6, x_3x_6, x_4x_6, x_3x_7, x_4x_7, x_5x_7, x_1x_8, x_4x_8, \\
&x_5x_8, x_6x_8, x_1x_9, x_2x_9, x_5x_9, x_6x_9, x_7x_9, x_1x_{10}, x_2x_{10}, x_3x_{10}, x_7x_{10}, x_8x_{10}, x_6x_{11}, \\
&x_7x_{11}, x_8x_{11}, x_9x_{11}, x_{10}x_{11}, x_1x_{12}, x_2x_{12}, x_3x_{12}, x_4x_{12}, x_5x_{12}, x_{11}x_{12}).
\end{aligned}
$$

This graph G has indMatch(G) $= 1$, and a complement clique complex isomorphic to the icosahedron. Its Stanley Reisner complex and Betti diagram have the following forms:



-	0	1	2	3	4	5	6	7	8	9
total:	1	36	160	327	412	412	327	160	36	1
0:	1	·	·	·	·	·	·	·	·	·
1:	·	36	160	315	300	112	12	·	·	·
2:	·	·	·	12	112	300	315	160	36	·
3:	·	·	·	·	·	·	·	·	·	1

Figure 5.3: Betti Table of the Stanley-Reisner Ideal of the Icosahedron

This satisfies $\text{Jump}(I_G) = [3; 2, 8]$ and $\text{relJump}(I_G) = [3; 2, 6]$.

Using $\text{Jump}(I_G)$, we provide a strengthening of Question 5.0.6:

Question 5.0.11. Given any strictly increasing sequence of positive integers a $=$ $[a_1, a_2, ..., a_{k-1}]$, does there exist a graph G such that

$$\text{Jump}(I_G) = [k; a_1, ..., a_{k-1}]?$$

This question is a strengthening of Question 5.0.6, as the former question can be rephrased in terms of this sequence. Note that if $\text{Jump}(I_G) = [k; a_1, ..., a_{k-1}]$, then $\text{reg}(I_G) = k + 1$.

Question 5.0.12 (Equivalent to Question 5.0.6)**.** [Open] Does there exist a constant N such that for all such $\text{Jump}(I_G) = [k; a_1, ..., a_{k-1}]$ with $a_1 \geq 2$, $k < N$?

For all jump sequences $\text{Jump}(I_G) = [k; a_1, a_2, ..., a_{k-1}]$ of edge ideals I_G, the sequence a_i is strictly increasing. This follows readily from an examination of the LCM lattice of I_G as seen in [9] as Theorem 5.2. In this thesis, we further restrict possible jump sequences $\mathrm{Jump}(I_G) = [k; a_1, ..., a_{k-1}]$ of edge ideals I_G. For example, we have the following restriction on jump sequences:

Theorem 5.0.13. Given any jump sequence $\mathrm{Jump}(I_G) = [k; a_1, a_2, ..., a_{k-1}]$,

$$2a_1 \leq a_2.$$

Example 5.0.14. This theorem states that no edge ideals exist with Betti diagrams of the form in Figure 5.4.

-	0	1	2	3	4	5	6	$\cdots$
total:	1	$*$	$*$	$*$	$*$	$*$	$*$	$\cdots$
0:	1	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
1:	$\cdot$	$*$	$*$	$\circ$	$\circ$	$\circ$	$\circ$	$\cdots$
2:	$\cdot$	$\cdot$	$\cdot$	$*$	$*$	$\circ$	$\circ$	$\cdots$
3:	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$*$	$*$	$\circ$	$\cdots$

-	0	1	2	3	4	5	6	$\cdots$
total:	1	$*$	$*$	$*$	$*$	$*$	$*$	$\cdots$
0:	1	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
1:	$\cdot$	$*$	$*$	$*$	$\circ$	$\circ$	$\circ$	$\cdots$
2:	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$*$	$*$	$\circ$	$\cdots$
3:	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\circ$	$\cdots$

Figure 5.4: Prohibited Betti Diagrams

We prove enumeratively that the following strengthening of Theorem 5.0.13.

Theorem 5.0.15 (Induced C_4-free Constraints)**.** Let G be a simple graph, I_G its edge ideal, and $\mathrm{Jump}(I_G) = [k; a_1, ..., a_{k-1}]$ its jump sequence. If $a_1 \geq 2$, then $a_2 \geq 8$.

In addition to enumerating these restrictions on permissible sequences, we provide several classes of edge ideals partially spanning this set of possible sequences. Characterizing the types of degree increases in the resolution of ideals of this form provides a tool to help characterize both the algebraic properties of edge ideals and the topological properties of certain flag simplicial complexes.

These algebraic questions are equivalent to a question about the topology of flag simplicial complexes:

Question 5.0.16. Given a flag simplicial complex Δ, and any ordering of the vertices $\{v_1, v_2, ..., v_n\}$. Let $V_i = \{v_1, v_2, ..., v_i\}$. Considering the chains of nested induced subcomplexes

$$\emptyset = \Delta|_{V_0} \subset \Delta|_{V_1} \subset \Delta|_{V_2} \subset \cdots \subset \Delta|_{V_k} \subset \cdots \subset \Delta|_{V_n},$$

characterizing sequences $a_i := \min\{k : \dim \widetilde{H}_i(V_k) \neq 0\} - i$ for $i \geq 1$ is equivalent to characterizing all jump sequences $\mathrm{Jump}(I_G)$. If the 1-skeleton of Δ is assumed to be C_4 free, what sequences $\{a_i\}_{i=1}^{\dim \Delta - 1}$ are possible?

5.1 Examples of Jump Sequences of I_G

Let G be a simple graph on vertex set $\{v_1, ..., v_n\}$ and I_G its edge ideal, $I_G \subseteq R = \Bbbk[x_1, ..., x_n]$. Letting $\mathcal{F}$ be a free resolution of I_G, we have that all Betti diagrams of edge ideals I_G are of the form in Figure 5.5.

-	0	1	2	3	4	$\cdots$	i	$\cdots$	n-2	n-1
total:	1	β_1	β_2	β_3	β_4	$\cdots$	β_i	$\cdots$	β_{n-2}	β_{n-1}
0:	1	·	·	·	·	·	·	·	·	·
1:	·	$\beta_{1,2}$	$\beta_{2,3}$	$\beta_{3,4}$	$\beta_{4,5}$	$\cdots$	$\beta_{i,i+1}$	$\cdots$	$\beta_{n-2,n-1}$	$\beta_{n-1,n}$
2:	·	·	$\beta_{2,4}$	$\beta_{3,5}$	$\beta_{4,6}$	$\cdots$	$\beta_{i,i+2}$	$\cdots$	$\beta_{n-2,n}$	·
$\vdots$	·	·	·							
k:	·	·	·	·	$\beta_{k,2k}$	$\cdots$	$\beta_{i,i+k}$	$\cdots$	·	·

Figure 5.5: Betti Table of an Edge Ideal

Remark 5.1.1. The first possible nonzero Betti number in each row is $\beta_{i,2i}$. This follows immediately from the Taylor resolution of the edge ideal, noting that all generators of I_G are of degree two. This gives us a staircase of sorts walking down the left edge of the Betti table, with each step of length at least one. On the other side of the Betti diagram, no Betti number can occur in degrees greater than n, so $\beta_{i,j} = 0$ for all $j > n$.

-	0	1	2	3	a_1+1	$\cdots$	a_2+1	$\cdots$	$a_{k-1}-1$	a_{k-1}	$a_{k-1}+1$
total:	1	β_1	β_2	$\cdots$	β_{a_1+1}	$\cdots$	β_{a_2+1}	$\cdots$	$\beta_{a_{k-1}-1}$	$\beta_{a_{k-1}}$	$\cdots$
0:	1	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	
1:	$\cdot$	$\beta_{1,2}$	$\beta_{2,3}$	$*$	$*$	$*$	$*$	$*$			
2:			a_1		β_{a_1+1,a_1+3}	$*$	$*$	$*$	$*$		
3:				a_2			β_{a_2+1,a_2+4}	$*$	$*$	$*$	$*$
$\vdots$									$*$	$*$	
k:						a_{k-1}					$\beta_{a_{k-1}+1,s}$

Figure 5.6: Betti Tables and Jump Sequences of I_G

So far, we have considered only Jump(I_G), rather than relJump(I_G). The advantage of working with the relative jump sequence relJump(I_G) is that the topological interpretation of the r_i is more straightforward than that of the a_i. If we have a subset of vertices $W \subset V$ of minimal size for which $\dim \widetilde{H}_{i-2}(\widehat{G^c}|_W) \neq 0$, then $r_i + 2$ is the number of vertices that must be added to W to find a set of vertices W' with $\dim \widetilde{H}_{i-1}(\widehat{G^c}|_{W'}) \neq 0$. It might be the case that a particular W has no subset of size $r_i + 2$ for which the rank of this homology is nonzero, but there exist at least *one* subset W such that we can find such a W'.

Example 5.1.2. The first example found with indMatch(G) $= 1$ and reg$(I_G) = 5$ was the complement graph of the 1-skeleton of the 600-cell [or hexacosichoron,

$\Delta_H]$ which we denote $G_H = (\Delta_H)_1^c$. This was first noted by Nevo and Peeva (private communication.) This is a graph on 120 vertices with 6420 edges. The corresponding edge ideal I_{G_H} has indMatch$(I_{G_H}) = 1$ and regularity 5, with jump sequence Jump$(I_{G_H}) = [3; 2, 8, 115]$.

Since Δ_H is equal to the clique closure of its 1-skeleton, and as its smallest induced cycle is of length 5, we have that $\Delta_H = \widehat{G_H^c}$ with induced matching size 1.

The smallest cycles are of length 5 and the smallest induced simplicial 2-spheres in Δ_H are icosahedron (one for each vertex of Δ_H, as an icosahedron formed of 20 tetrahedral cells lie around each vertex.) Finally, the entire complex is a simplicial 3-sphere on 120 vertices. So our number of vertices involved in the minimal reduced homology generators are 5, 12, and 120 respectively – giving rise to jump sequence a $= [3; 2, 8, 115]$.

5.2 Jump Sequences and Bounds on Regularity of I_G

We now prove the first of the theorems restricting jump sequences Jump(I_G).

Theorem 5.2.1. Let I_G be an edge ideal with jump sequence Jump$(I_G) = [k; a_1, a_2, ..., a_{k-1}]$ and relative jump sequence relJump$(I_G) = [k; r_1, r_2, ..., r_{k-1}]$. Then the following equivalent statements hold:

(i) $2a_1 \leq a_2$

(ii) $r_1 \leq r_2$.

It is convenient to reduce this problem for general jump sequences to a problem on jump sequences of length 2. The following lemma allows us to go even further:

Lemma 5.2.2 (Subgraph Reduction Lemma)**.** Let G be a graph such that I_G has jump sequence $[k; a_1, a_2, ..., a_n]$, with $k \geq 3$. There exists an induced subgraph H of I_G on vertex set $W \subset V$ of size $|W| = a_2 + 4$ such that I_H has jump sequence $[3; a_1', a_2]$, and $a_1 \leq a_1'$, and such that H has no induced subgraphs $W' \subseteq W$ such that $\dim \widetilde{H}_2((\Delta_H)|_{W'}, k) \neq 0$.

Proof. This follows from the definition of our jump sequences,

$$a_r = \min\{i : \beta_{i,j} \neq 0 \text{ and } j - i = r\} - 1.$$

So a_2 is the smallest Betti index such that $\beta_{a_2+1, a_2+4} \neq 0$. Via Theorem 3.2.9, this is equivalent to saying there exists some subset of vertices $W \subset V$ such that $|W| = a_2 + 4$ with the dimension of the reduced second homology of $(\Delta_G)|_W = (\Delta_{G|_W})$ nonzero, and no smaller subset $W' \subset V$ will give us nonzero $\widetilde{H}_2$. Let $\mathcal{W}$ denote the set of all such $W \subseteq V$.

Among such subsets W, choose one such that the size of the smallest induced cycle in $(\Delta_G)|_W$, i.e. given any pair $W, W' \in \mathcal{W}$, if c is an induced cycle of minimal size in Δ_W, then there exists a cycle $c' \in W'$ in $\Delta|_{W'}$ such that the $|c'| \leq |c|$. Let H be our induced subgraph of G on vertex set W.

By construction, the minimal cycles of G^c are of length smaller than or equal to the minimal cycles of H^c, and the minimal vertex sets of G on which induced subcomplexes of $\widehat{G^c}$ have $\widetilde{H}_2 \neq 0$ are of the same size as those of $\widehat{H^c}$, so for G we have the jump sequence of I_G is $[k; a_1, a_2, ..., a_{k-1}]$ and the jump sequence of I_H is $[3; a'_1, a_2]$ with $a_1 \leq a'_1$. $\square$

The upshot of Lemma 5.2.2 is the following: If we can prove the Theorem 5.2.1 holds for graphs of the form H, $|H| = a_2 + 5$, with jump sequence $a = [3; a'_1, a_2]$, we will have that it holds for all graphs G, using the inequalities $2a_1 \leq 2a'_1 \leq a_2$. We can now reduce the general problem to cases such that the entire Stanley-Reisner complex of the graph is our minimal generator of nonzero $\widetilde{H}_2$, and removing any vertex $v \in H$ will drop the dimension of the homology by at least one.

Proof of Theorem 5.2.1. Without loss of generality, assume G is a graph of the form introduced in Lemma 5.2.2. If $a_1 = 1$, then $a_2 \geq 2$ since I_G is generated in degree 2. So we consider only the case where $a_1 \geq 2$. We note that in this case, the size of the vertex set of G is $|V| = a_2 + 4$.

Assume by contradiction that $2a_1 > a_2$, and let W be the vertex set minimal induced cycle $c \in G^c$. We have that the length of $c = |W| = a_1 + 3$. Choose two nonadjacent vertices $\{v, w\}$ in this cycle c. Orient the cycle from v to w, then back to v, and partition the vertices in c into sets $V_1, V_2 \subset W$, such that

$$V_1 = \{v, v_1, v_2, ..., v_k, w\}$$

$$V_2 = \{w, w_1, w_2, ..., w_{a_1-k+1}, v\}$$

where $\{v, v_1, v_2, ..., v_k, w\}$ and $\{w, w_1, w_2, ..., w_{a_1-k+1}, v\}$ are vertices in the oriented paths from v to w, then from w to v, endpoints inclusive. Let $K = V \backslash W$, and $K_1 = K \cup V_1$, $K_2 = K \cup V_2$. We consider the complexes $\Delta_{K_1} = (\Delta_G)|_{K_1}$ and

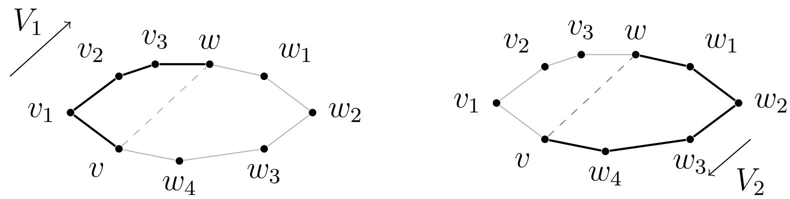


Figure 5.7: Partitioning Induced Cycle into V_1 and V_2

$\Delta_{K_2} = (\Delta_G)\,|_{K_2}$, on these vertex sets K_1 and K_2. We also let $\Delta_{K'} = \Delta_{K_1} \cap \Delta_{K_2}$, and $K' = K_1 \cap K_2 = K \cup \{v, w\}$.

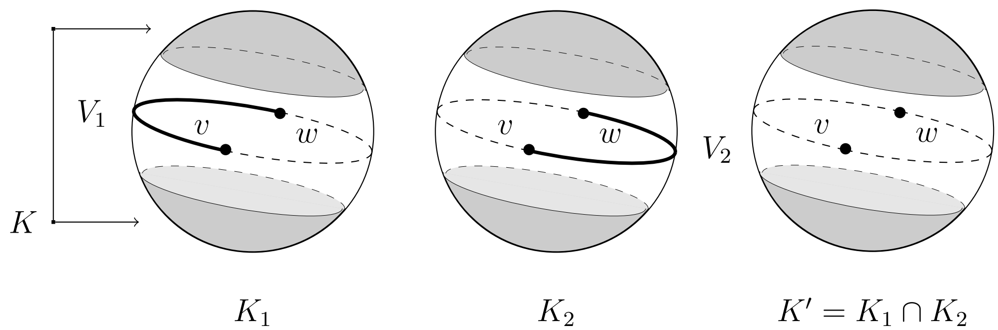


Figure 5.8: Covering Complex Δ with Δ_{K_1} and Δ_{K_2}

Lemma 5.2.3. We have that

1. $\Delta_{K'} = \Delta_{K_1} \cap \Delta_{K_2} = \Delta_{K_1 \cap K_1} = (\Delta_G)\,|_{K'}$ and
2. $\Delta_G = \Delta_{K_1} \cup \Delta_{K_2}$.

Proof. The first fact follows immediately from properties of induced subcomplexes, as for general simplicial complexes Δ with sets of vertices $S, T \subseteq V$, we have

$$\Delta|_{S \cap T} = \Delta|_S \cap \Delta|_T.$$

The second is not true for all simplicial complexes. For complexes Δ of the form above, we wish to show that every $\sigma \in \Delta$ is in either the induced subcomplex Δ_{K_1} or Δ_{K_2}. It is sufficient to show this for edges of Δ, as clique closure of Δ finishes the argument. It is clear that any edges on vertices entirely contained in $K_1, or K_2$ are in $\Delta_{K_1} \cup \Delta_{K_2}$. An edge running between the two (but not contained fully in either) would have to be of the form $\{v_i, w_j\}$ for $v_i \in V_1, w \in V_2$. However, by construction of c as a minimal cycle, no such chords exist. The result follows. $\square$

With this characterization in hand, we use a Mayer-Vietoris sequence (as in Theorem 3.2.10) to finish the proof of Theorem 5.2.1.

$$\cdots \to H_2(\Delta_{K_1}) \oplus H_2(\Delta_{K_2}) \to H_2(\Delta_G) \xrightarrow{\partial} H_1(\Delta_{K'}) \to H_1(\Delta_{K_1}) \oplus H_1(\Delta_{K_2}) \to \cdots$$

As we had assumed G had no proper induced subgraphs G' with $H_2(\Delta_{G'}) \neq 0$, we have the leftmost term is zero. So the map $\partial : H_2(\Delta_G) \to H_1(\Delta_{K'})$ is injective, and $\dim H_1(\Delta_{K'}) \neq 0$. This subset K' is of size

$$|K| + 2 = |V| - |W| + 2 = (a_2 + 4) - (a_1 + 3) + 2 = a_2 - a_1 + 3.$$

By assumption, $2a_1 > a_2$, we have that $|K'| < 2a_1 - a_1 + 3 = a_1 + 3$. So $|K'|$ is a set of vertices strictly smaller than those of c generating a nonzero first homology, which gives us our desired contradiction. $\square$

Remark 5.2.4. Returning to the language introduced in Question 5.0.16, we rephrase Theorem 5.2.1.

Let Δ be any flag complex on n vertices with $\dim H_2(\Delta) \neq 0$ and $\dim H_2(\Delta \backslash v) = 0$ for any vertex $v \in \Delta$. Given any ordering of the vertices of Δ, $V = \{v_1, ..., v_n\}$, we will let $V_i := \{v_1, ..., v_i\}$. Consider the chain of inclusions of the induced simplicial complexes,

$$\emptyset = \Delta|_{V_0} \subset \Delta|_{V_1} \subset \Delta|_{V_2} \subset \cdots \subset \Delta|_{V_k} \subset \cdots \subset \Delta|_{V_n}.$$

Across all such chains, choose one such that k, the first index for which

$$\dim H_1(\Delta|_{V_k}) \neq 0,$$

is minimal. Then we have the total number of vertices in our complex must be at least 2k.

Example 5.2.5. It is not the case that all minimal homology generators of $\widetilde{H}_2$ can be chosen to be spheres, or that given any generator K of nonzero $\widetilde{H}_1$ that the vertices can be partitioned into hemispheres K_1 and K_2 in such a way that $K_1 \cap K_2 = K$ and $K_1 \cap K_2 = \Delta_G$. The 1-skeleton of the following complex $\widehat{G^c}$ is an example of when this partitioning fails. If the edges in bold are cho-

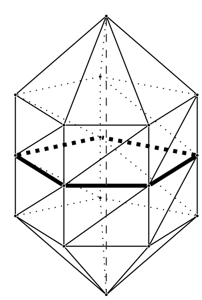


Figure 5.9: Nonsphere Complex Δ, no $\widetilde{H}_2(\Delta') \neq 0$ for Induced $\Delta' \subset \Delta$

sen as the minimal generator of first homology, there is no way of partitioning the remaining vertices into hemispheres such that the upper and lower hemisphere intersect in this cycle, and the union of the induced complexes on the hemispheres is the entire complex. The edge running from the top vertex to the bottom vertex will not be contained in the induced subcomplexes.

5.3 Constraints on Betti Tables

Theorem 5.2.1 immediately gives that if $\beta_{2,4}(I_G) = 0$, then $\beta_{i,i+3}(I_G) = 0$ for $i \leq 4$. To restate, we have the following Corollary:

Corollary 5.3.1. Let G be a simple graph, I_G its edge ideal, and $\text{Jump}(I_G) = [k; a_1, ..., a_{k-1}]$ its jump sequence. If $a_1 = 2$ then $a_2 \geq 4$.

Theorem 5.2.1 shows that notable constraints exist on the type of sygyzies found in edge ideals of graphs with C_4-free complement. Similarly, if $\beta_{3,5}(I_G) = 0$, then $\beta_{i,i+3}(I_G) = 0$ for $i \leq 6$. Hence, no edge ideals exist with Betti diagrams of the forms:

-	0	1	2	3	4	5	6	$\cdots$
total:	1	$*$	$*$	$*$	$*$	$*$	$*$	$\cdots$
0:	1	·	·	·	·	·	·	·
1:	·	$*$	$*$	$\circ$	$\circ$	$\circ$	$\circ$	$\cdots$
2:	·	·	·	$*$	$*$	$\circ$	$\circ$	$\cdots$
3:	·	·	·	·	$*$	$*$	$\circ$	$\cdots$

-	0	1	2	3	4	5	6	$\cdots$
total:	1	$*$	$*$	$*$	$*$	$*$	$*$	$\cdots$
0:	1	·	·	·	·	·	·	·
1:	·	$*$	$*$	$*$	$\circ$	$\circ$	$\circ$	$\cdots$
2:	·	·	·	·	$*$	$*$	$\circ$	$\cdots$
3:	·	·	·	·	·	·	$\circ$	$\cdots$

Figure 5.10: Prohibited Betti Diagrams: $\beta_{2,4} = 0$

This lower bound in Corollary 5.3.1 on vanishing $\beta_{i,i+3}(I_G)$ is not sharp. We extend the results in Theorem 5.2.1 to the following theorem:

Theorem 5.0.15 (Induced C_4-free Constraints)**.** Let G be a simple graph, I_G its edge ideal, and $\text{Jump}(I_G) = [k; a_1, ..., a_{k-1}]$ its jump sequence. If $a_1 \geq 2$, then $a_2 \geq 8$.

We first introduce a definition.

Definition 5.3.2. A *vertex-induced minimal reduced r-homology* (r-VMH) complex Δ is a simplicial complex on vertex set V such that

(i) $\dim \widetilde{H}_r(\Delta) \neq 0$

(ii) $\dim \widetilde{H}_r(\Delta|_W) = 0$

for all proper subsets of vertices $W \subseteq V$.

To prove Theorem 5.0.15, we need to characterize the minimal size of 2-VMH flag complexes.

Proposition 5.3.3. A 2-VMH flag complex Δ on n vertices which has an induced C_4-free 1-skeleton must satisfy:

(i) Δ_1 is connected,

(ii) $(\Delta_1)^c$ is connected, and

(iii) for all $v \in \Delta$, $\deg(v) \geq 5$.

Proof. Proof of (i): If Δ_1 is not connected, Δ is disconnected. Hence, $\Delta = A \cup B$ for subcomplexes A and B with either $\widetilde{H}_2(A) \neq 0$ and $\widetilde{H}_2(B) \neq 0$. So $A \subseteq \Delta$ or $B \subseteq \Delta$ is a vertex induced subcomplex generating nonzero second homology.

Proof of (ii): If $(\Delta_1)^c$ is not connected, then $(\Delta_1)^c = A \cup B$ for some complexes A, B. Choose $e \in A$ and $e' \in B$. As A, B are disconnected in $(\Delta_1)^c$, $\{e, e'\}$ are induced disjoint sets. By Corollary 4.3.4, this implies that Δ is not induced C_4-free.

Proof of (iii): We divide our cases up by vertex degrees. Throughout, let W denote the set of vertices of Δ excluding a fixed vertex $v \in V$, i.e. $W = V/\{v\}$.

($\deg(v) = 0$): This implies an isolated vertex, contradicting part (i).

($\deg(v) = 1$): If $v \in V$ is of degree one, then the complex Δ and $\Delta|_W$ are homeomorphic via contraction along the edge attached to v.

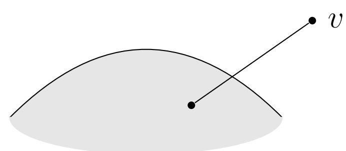
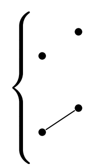


Figure 5.11: $\deg(v) = 1$ case

($deg(v) = 2$): If $v \in V$ is of degree 2, then the link of v is one of the following:

link$(v) =$ 

As seen in Figure 5.12, in Case , we have that no 2-simplices contain v. In Case , $\Delta \cong \Delta|W$. Hence, in both cases we see that $\widetilde{H}_2(\Delta) = \widetilde{H}_2(\Delta|W)$.

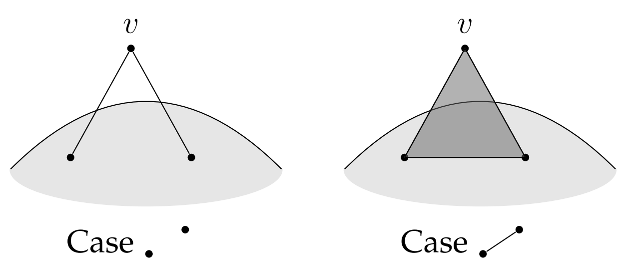
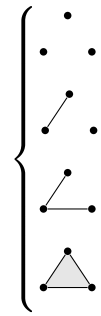
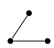
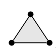


Figure 5.12: $\deg(v) = 2$ case

($\deg(v) = 3$): If $v \in V$ is of degree 3, then the link of v is one of the following:

link$(v) =$ 

We illustrate all degree 3 cases in Figure 5.13. In Case , we have that $\Delta|_{\{v\} \cup\, n(v)}$ is a claw, so no 2-simplices contain v. In Case , we have that $\Delta \cong \Delta|_W \vee S^1$. In Case  and Case , we have that $\Delta \cong \Delta|_W$.

In each of these cases, $\widetilde{H}_2(\Delta) = \widetilde{H}_2(\Delta|_W)$.

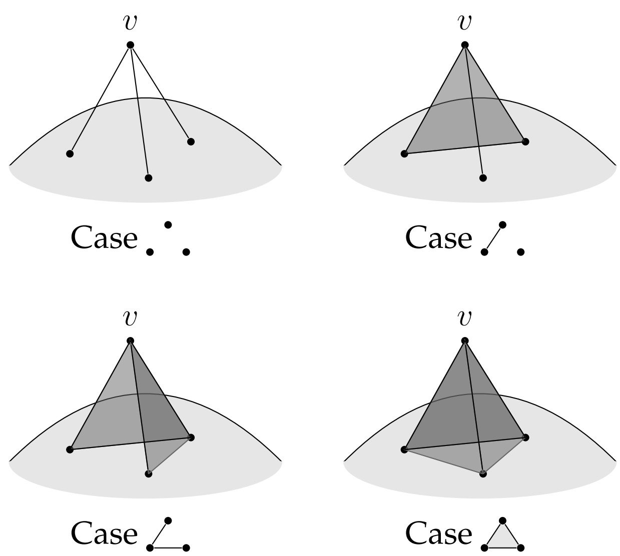
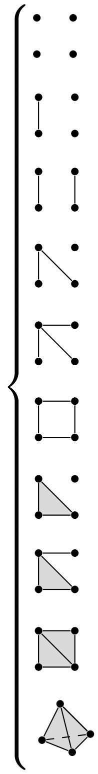


Figure 5.13: $\deg(v) = 3$ case

($\deg(v) = 4$): If $v \in V$ is of degree 4, then the link of v is one of the following:

link$(v) =$ 

We illustrate all degree 4 cases in Figure 5.14.

In Case , link(v) is four disjoint points. So no 2-simplices in Δ contain v and $\widetilde{H}_2(\Delta) = \widetilde{H}_2(\Delta|_W)$. In Case , we have that $\Delta \cong \Delta|_W \vee S^1 \vee S^1$. So $\widetilde{H}_2(\Delta) = \widetilde{H}_2(\Delta|_W)$. In Cases , , and , we have that $\Delta \cong \Delta|_W \vee S^1$. So $\widetilde{H}_2(\Delta) = \widetilde{H}_2(\Delta)$. Case violates C_4-freeness of Δ.

All of the remaining cases satisfy $\Delta \cong \Delta|_W$ via a chain of simplicial collapses. Hence, $\widetilde{H}_2(\Delta) = \widetilde{H}_2(\Delta|_W)$. So for each vertex with $\deg(v) \leq 4$, we have that $\widetilde{H}_2(\Delta) = \widetilde{H}_2(\Delta|_W)$.

This completes the proof of Proposition 5.3.3. $\square$

Using this, we can restrict our search to edge ideals of graphs which are connected and complement connected, with vertices of degree at most $n - 6$.

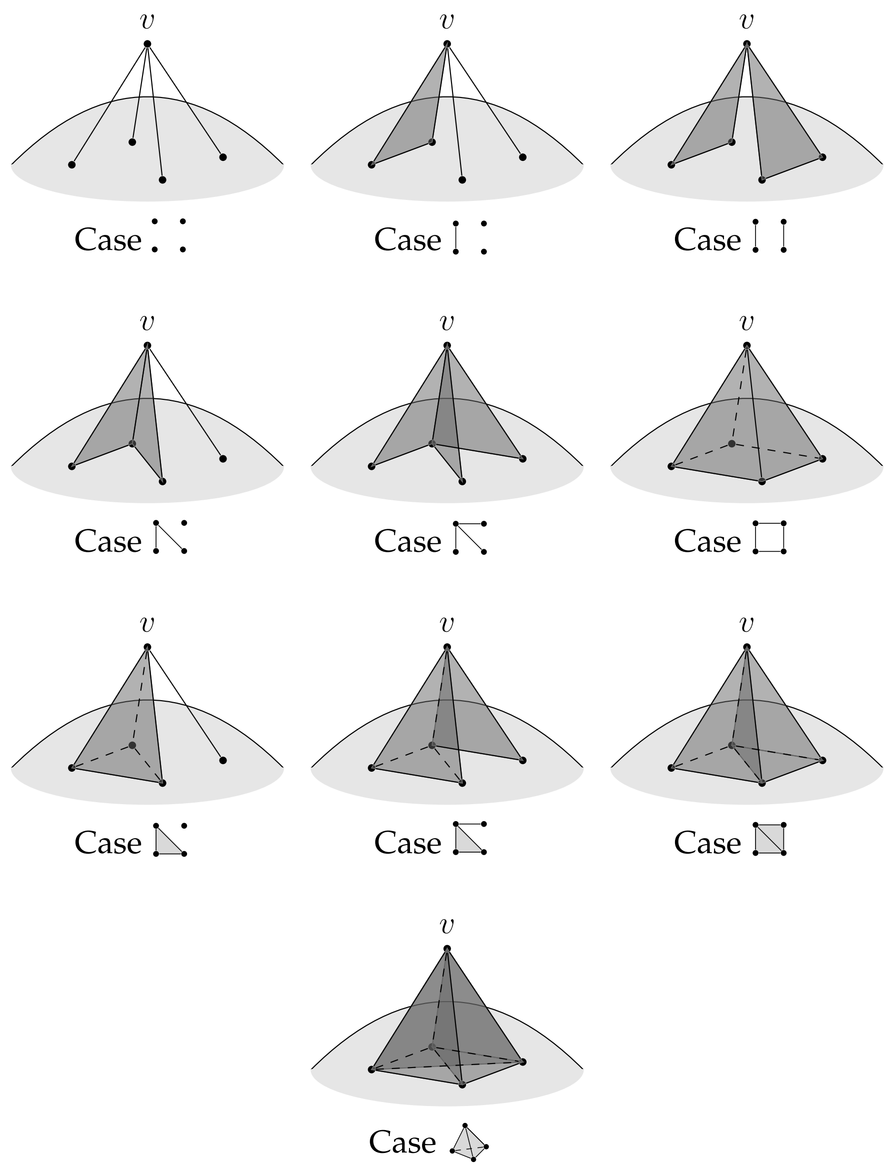


Figure 5.14: $\deg(v) = 4$ case

We used the Nauty [24] interface for Macaulay2 [21] and searched for all graphs up to isomorphism which are connected and have all vertices of degree at least 5. This produces fairly short lists for graphs of vertex size n=8 through 11.

In practice, enumerating all connected graphs on n vertices with vertex de-

n	non-isom. G	conn. complement, $\deg(v) \leq n-6$	$\mathrm{reg}(I_G) = 3$	$\mathrm{reg}(I_G) = 4$
8	12346	2	0	0
9	274668	531	0	0
10	12005168	89402	1	0
11	1018997864	21603340	11	0

Table 5.1: Number of graphs G on n vertices, filtered by type.

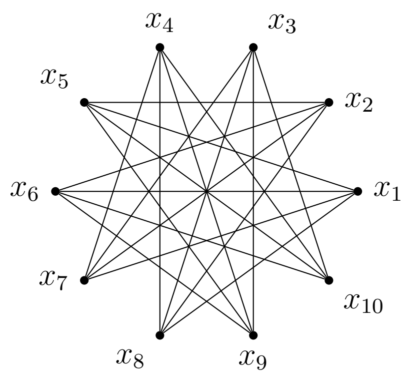
gree at least 5, complementing, then filtering for connectedness in Nauty is computationally too expensive to be feasible. The complementation routines in Nauty are more time consuming than taking the resolution of an edge ideal I_G itself – so to avoid this, we simply enumerated all graphs on n vertices with at most $n - 6$ vertices.

As seen in the table, extensive graph lists are pruned to much more manageable size. We then compute the edge ideals from the Sparse6 string produced by Nauty and take their resolutions, filtering for those with $\beta_{2,4}(I_G) = 0$. With Proposition 5.3.3 in hand, this is sufficient to complete the proof of Theorem 5.0.15.

As the lists of graphs which are connected, induced C_4-free, have nonlinear resolutions, and have maximal vertex degree $n - 6$ are so short for $n = 10$ and $n = 11$, we include them here.

5.3.1 Case 1: n=10 vertex case

Out of 89,402 connected graphs on $n = 10$ vertices with degree at most $n - 6 = 4$ vertices, exactly one has a nonlinear resolution and $\mathrm{indMatch}(G) = 1$.

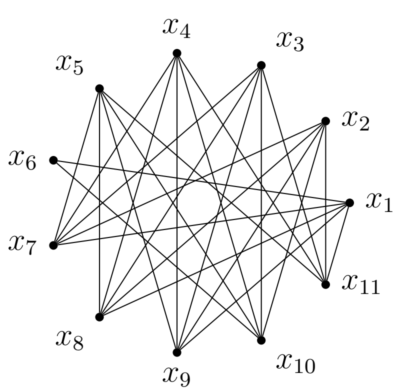


-	0	1	2	3	4	5	6	7	8
total:	1	20	60	97	110	85	40	10	1
0:	1	·	·	·	·	·	·	·	·
1:	·	20	60	65	30	5	·	·	·
2:	·	·	·	32	80	80	40	10	1

Figure 5.15: 10 Vertex Graph

5.3.2 Case 2: n=11 vertex case

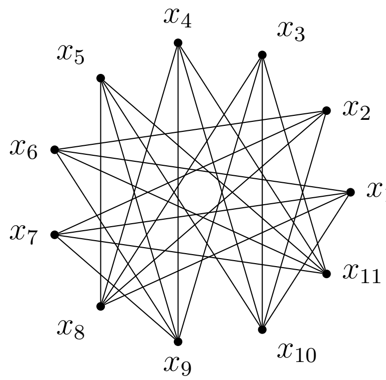
Out of 21,603,340 connected graphs on $n = 11$ vertices with degree at most $n - 6 = 5$ vertices, exactly 11 have nonlinear resolutions and indMatch$(G) = 1$.



-	0	1	2	3	4	5	6	7	8	9
total:	1	26	101	200	250	210	120	45	10	1
0:	1	·	·	·	·	·	·	·	·	·
1:	·	26	101	184	202	142	64	17	2	·
2:	·	·	·	16	48	68	56	28	8	1

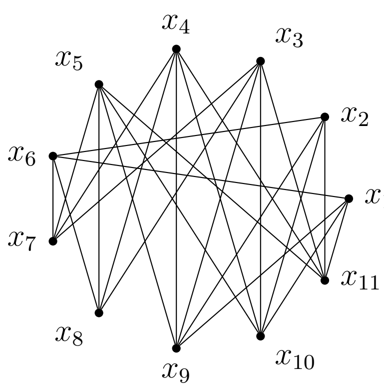
Figure 5.16: 11 Vertex Graph 1

For an edge ideal I_G, the location of the first nonlinear syzygy is independent of the characteristic of the coefficient field. Hence, it suffices to calculate the integral homologies of the Stanley-Reisner complexes of I_G, a computation we



-	0	1	2	3	4	5	6	7	8	9
total:	1	24	82	157	215	207	131	51	11	1
0:	1	·	·	·	·	·	·	·	·	·
1:	·	24	82	109	71	23	3	·	·	·
2:	·	·	·	48	144	184	128	51	11	1

Figure 5.17: 11 Vertex Graph 2



-	0	1	2	3	4	5	6	7	8	9
total:	1	25	90	173	223	198	118	45	10	1
0:	1	·	·	·	·	·	·	·	·	·
1:	·	25	90	137	115	57	16	2	·	·
2:	·	·	·	36	108	141	102	43	10	1

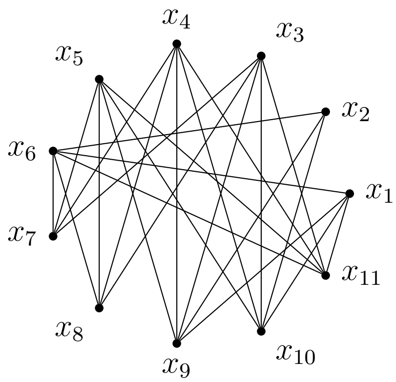
Figure 5.18: 11 Vertex Graph 3

perform in Gap.

The code used to prove Theorem 5.0.15 can be found in Appendix A.

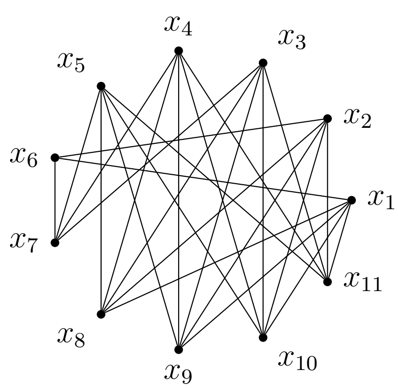
5.4 Constraints on Betti Tables

So if $\beta_{2,4}(I_G) = 0$, all Betti diagrams lie above the line in Figure 5.27.



-	0	1	2	3	4	5	6	7	8	9
total:	1	25	90	167	208	184	112	44	10	1
0:	1	·	·	·	·	·	·	·	·	·
1:	·	25	90	137	109	48	11	1	·	·
2:	·	·	·	30	99	136	101	43	10	1

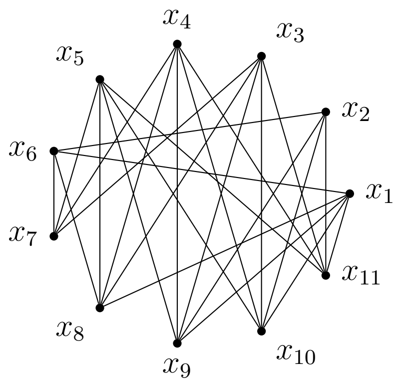
Figure 5.19: 11 Vertex Graph 4



-	0	1	2	3	4	5	6	7	8	9
total:	1	26	99	195	246	209	120	45	10	1
0:	1	·	·	·	·	·	·	·	·	·
1:	·	26	99	171	174	111	44	10	1	·
2:	·	·	·	24	72	98	76	35	9	1

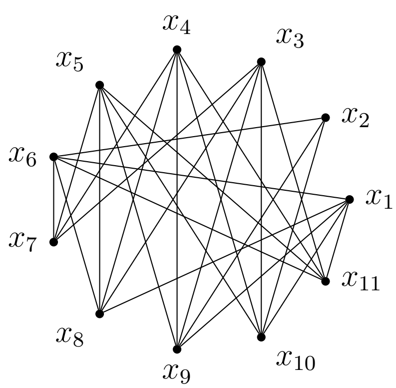
Figure 5.20: 11 Vertex Graph 5

This bounds the increase in degrees of the sygygies of I_G in terms of minimal cycle length in G^c. It is not however the case that for all graphs, these relative jump sequences (i.e. the lengths of the *stairs* of the lower edge of the resolution) must be weakly increasing. In Example 6.3.5 in Section 6.3 we construct such a counterexample. We also provide a general algorithm for constructing large classes of Betti diagrams of edge ideals. It should be noted that the number of vertices involved in the example are high - and no graphs of smaller size are



-	0	1	2	3	4	5	6	7	8	9
total:	1	26	97	185	231	200	118	45	10	1
0:	1	·	·	·	·	·	·	·	·	·
1:	·	26	97	158	141	74	22	3	·	·
2:	·	·	·	27	90	126	96	42	10	1

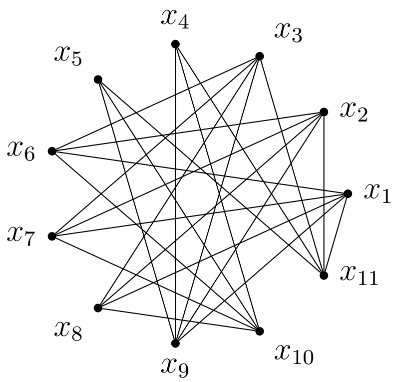
Figure 5.21: 11 Vertex Graph 6



-	0	1	2	3	4	5	6	7	8	9
total:	1	26	97	182	222	190	113	44	10	1
0:	1	·	·	·	·	·	·	·	·	·
1:	·	26	97	158	138	68	18	2	·	·
2:	·	·	·	24	84	122	95	42	10	1

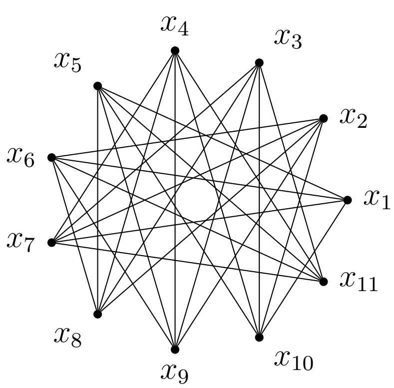
Figure 5.22: 11 Vertex Graph 7

currently known whose edge ideals exhibit this behavior.



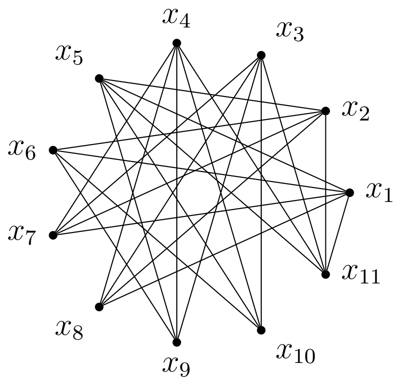
-	0	1	2	3	4	5	6	7	8	9
total:	1	24	84	158	203	183	112	44	10	1
0:	1	·	·	·	·	·	·	·	·	·
1:	·	24	84	122	95	42	10	1	·	·
2:	·	·	·	36	108	141	102	43	10	1

Figure 5.23: 11 Vertex Graph 8



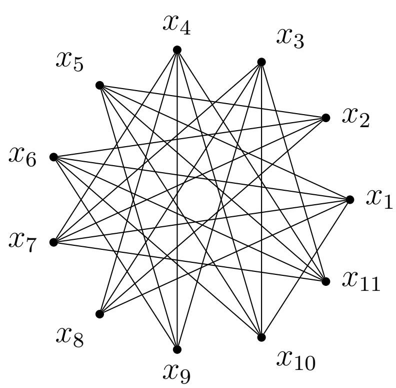
-	0	1	2	3	4	5	6	7	8	9
total:	1	27	101	182	219	197	125	50	11	1
0:	1	·	·	·	·	·	·	·	·	·
1:	·	27	101	160	120	43	6	·	·	·
2:	·	·	·	22	99	154	119	50	11	1

Figure 5.24: 11 Vertex Graph 9



-	0	1	2	3	4	5	6	7	8	9
total:	1	25	88	164	215	204	130	51	11	1
0:	1	·	·	·	·	·	·	·	·	·
1:	·	25	88	124	83	26	3	·	·	·
2:	·	·	·	40	132	178	127	51	11	1

Figure 5.25: 11 Vertex Graph 10



-	0	1	2	3	4	5	6	7	8	9
total:	1	26	94	171	215	201	129	51	11	1
0:	1	·	·	·	·	·	·	·	·	·
1:	·	26	94	139	95	29	3	·	·	·
2:	·	·	·	32	120	172	126	51	11	1

Figure 5.26: 11 Vertex Graph 11

-	0	1	2	3	4	5	6	7	8	···
total:	1	∗	∗	∗	∗	∗	∗	∗	∗	···
0:	1	·	·	·	·	·	·	·	·	·
1:	·	∗	∗	○	○	○	○	○	○	···
2:	·	·	0	∗	∗	∗	∗	∗	∗	○
3:	·	·	·	·	·	·	·	·	·	∗

Figure 5.27: Theorem 5.0.15

CHAPTER 6

CLASSES OF EDGE IDEALS WITH HIGH REGULARITY

6.1 Joins and Products

Definition 6.1.1. Let K, L be simplicial complexes on disjoint vertex sets. The *combinatorial join of K and L* is

$$K * L := \{S \cup T : S \in K, T \in L\}.$$

Example 6.1.2. If K is an edge and L is a vertex, then $K * L$ is a 2-simplex,

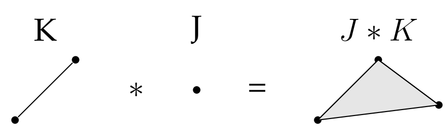
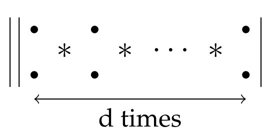


Figure 6.1: Join $J * K$ of Simplicial Complexes J, K

Some special cases of combinatorial join of note:

1. If L is a point, say $L = \{x\}$, and K any simplicial complex, then $K * L =$ cone(K, x).
2. The join of two points, iterated d times, is isomorphic to S^{d-1}, i.e.



$$\cong S^{d-1}$$

$$\cong \partial\,(\text{cross polytope}) = \partial(\text{conv}\{\pm e_i\}_1^d)$$

6.2 Betti Numbers of $\Delta_1 \uplus \Delta_2$

We compute the ideals of complexes $\Delta_1 \uplus \Delta_2$ and the Betti numbers of their resolutions. In a previous paper [39], we produced this formula specifically in the case of edge ideals. We include the more general version here.

Proposition 6.2.1. Let Δ_1 and Δ_2 be two Stanley-Reisner complexes respectively on vertex sets $\{v_1, ..., v_n\}$ and $\{w_1, ..., w_m\}$, with Stanley-Reisner ideals $I_{\Delta_1} \subseteq \Bbbk[x_1, ..., x_n]$ and $I_{\Delta_2} \subseteq \Bbbk[y_1, .., y_m]$. Then $\Delta_1 \uplus \Delta_2$ has Stanley-Reisner ideal

$$I_{\Delta_1 \uplus \Delta_2} = I_{\Delta_1} + I_{\Delta_2} + (x_i y_j : i = 1, ..., n, j = 1, ..., m) \subseteq \Bbbk[x_1, ..., x_n, y_1, ..., y_m]$$

Note that in the case of I_{Δ_1} and I_{Δ_2} edge ideals, then $I_{\Delta_1 \uplus \Delta_2}$ is again an edge ideal. We note that the Betti numbers of this complex can be computed via Hochster's formula in terms of sums of the Betti numbers of our original complexes as follows:

Proposition 6.2.2. Given two square-free ideals $I_{\Delta_1} \subset \Bbbk[x_1, ..., x_s]$ and $I_{\Delta_2} \subset \Bbbk[y_1, ..., y_t]$ with Stanley-Reisner complexes Δ_1 and Δ_2 respectively, the ideal $I_{\Delta_1 \uplus \Delta_2}$ has Betti numbers

$$\begin{aligned}\beta_{i,i+1}(I_{\Delta_1 \uplus \Delta_2}) = \beta_{i,i+1}(I_{\Delta_1}) + \beta_{i,i+1}(I_{\Delta_2}) \\ + \sum_{j=1}^{i} \left(\binom{m}{i-j+1} \beta_{j-1,j}(I_{\Delta_1}) + \binom{n}{j} \beta_{i-j,i-j+1}(I_{\Delta_2}) \right) \\ + \binom{m+n}{i+1} - \binom{m}{i+1} - \binom{n}{i+1}.\end{aligned}$$

For terms in the nonlinear strands, we have for $s \geq 2$,

$$\beta_{i,i+s}(I_{\Delta_1 \cup \Delta_2}) = \beta_{i,i+s}(I_{\Delta_1}) + \beta_{i,i+s}(I_{\Delta_2}) \\ + \sum_{j=1}^{i+s-1} \left(\binom{m}{i-j+s} \beta_{j-s,j}(I_{\Delta_1}) + \binom{n}{j} \beta_{i-j,i-j+s}(I_{\Delta_2}) \right).$$

Proof of Proposition 6.2.2. Using Hochster's formula, we rewrite $\beta_{i,i+s}(I_{\Delta_1 \cup \Delta_2})$ in terms of the dimensions of the homologies of sets of size $i+s$. For terms in the linear strand (for which s=1) this becomes:

$$\begin{aligned}\beta_{i,i+1}(I_{\Delta_1 \cup \Delta_2}) &= \sum_{|W|=i+1} \widetilde{H}_0 \left(\Delta_1 \cup \Delta_2|_W \right) \\ &= \sum_{\substack{|W|=i+1 \\ W \subseteq V_1}} \widetilde{H}_0 \left(\Delta_1|_W \right) + \sum_{\substack{|W|=i+1 \\ W \subseteq V_2}} \widetilde{H}_0 \left(\Delta_2|_W \right) \\ &\quad + \sum_{\substack{|R|+|S|=i+1 \\ R \subseteq V_1,\ |R|=j \\ S \subseteq V_2,\ |S|=i-j+1}} \left[\widetilde{H}_0 \left(\Delta_1|_R \right) + \widetilde{H}_0 \left(\Delta_2|_S \right) + 1 \right]\end{aligned}$$

The extra 1 in the rightmost summand corrects the count for reduced homology of the two subsets. The first two terms in the summand are the Betti numbers of the original ideals. We rewrite the sum using this, with $R \subseteq V_1$ and $S \subseteq V_2$, then sum across all subsets with the appropriate counts:

$$
\begin{aligned}
\beta_{i,i+1}(I_{\Delta_1 \cup \Delta_2}) &= \beta_{i,i+1}(I_{\Delta_1}) + \beta_{i,i+1}(I_{\Delta_2}) \\
&\quad + \sum_{j=1}^{i} \sum_{|S|=i-j+1} \left(\sum_{|R|=j} \left[\widetilde{H}_0\left(\Delta_1|_R\right) + \widetilde{H}_0\left(\Delta_2|_S\right) + 1 \right] \right) \\
&= \beta_{i,i+1}(I_{\Delta_1}) + \beta_{i,i+1}(I_{\Delta_2}) \\
&\quad + \sum_{j=1}^{i} \sum_{|S|=i-j+1} \left(\beta_{j-1,j}(I_{\Delta_1}) + \binom{n}{j} \widetilde{H}_0\left(\Delta_2|_S\right) + \binom{n}{j} \right) \\
&= \beta_{i,i+1}(I_{\Delta_1}) + \beta_{i,i+1}(I_{\Delta_2}) \\
&\quad + \sum_{j=1}^{i} \sum_{|S|=i-j+1} \left(\beta_{j-1,j}(I_{\Delta_1}) + \binom{n}{j} \widetilde{H}_0\left(\Delta_2|_S\right) + \binom{n}{j} \right) \\
&= \beta_{i,i+1}(I_{\Delta_1}) + \beta_{i,i+1}(I_{\Delta_2}) \\
&\quad + \sum_{j=1}^{i} \left(\binom{m}{i-j-1} \beta_{j-1,j}(I_{\Delta_1}) + \binom{n}{j} \beta_{i-j,i-j+1} \right) \\
&\quad + \sum_{j=1}^{i} \left(\binom{m}{i-j+1} \binom{n}{j} \right)
\end{aligned}
$$

The final Betti number count above uses the combinatorial identity

$$
\begin{aligned}
\sum_{j=1}^{i} \binom{m}{i-j+1} \binom{n}{j} &= \left[\sum_{j=0}^{i+1} \binom{m}{i-j+1} \binom{n}{j} \right] - \binom{m}{i+1} - \binom{n}{i+1} \\
&= \binom{m+n}{i+1} - \binom{m}{i+1} - \binom{n}{i+1}
\end{aligned}
$$

This finishes the proof for the calculation of Betti numbers in the linear strand, producing the formula above. The count for the Betti numbers $\beta_{i,i+s}$ in the non-linear strands is identical, removing the binomial coefficient terms coming from the reduced homology zero correction. □

It was noted in [27] that the argument used in our early paper [39], which expressed the Betti numbers of $\Delta_G \cup \Delta_H$ for edge ideals I_G and I_H proved this more general statement.

6.3 Corner Diagrams and Achievable Jump Sequences

In this section, we describe a technique of producing relative jump sequences which weakly increasing, as it appears all relative jump sequences of edge ideals of flag polytopes are. Specifically, we construct a counterexample to all relative jump sequences being weakly increasing. Throughout, we will refer to *Betti diagrams of shape* $\mathbf{a} = [k; a_1, ..., a_{k-1}]$, or $\mathcal{B}_{\mathbf{a}}$, the set of all Betti diagrams of edge ideals with jump sequence $\mathbf{a} = \text{Jump}(I_G)$.

Definition 6.3.1. The *corner sum of two jump sequences* $\mathbf{a} = [k; a_1, a_2, ..., a_{k-1}]$ and $\mathbf{b} = [l; b_1, b_1, ..., b_{k-1}, b_j, ..., b_{l-1}]$, where $k \leq l$, we define to be

$$[l; c_1, c_2, ..., c_{k-1}, b_j, ..., b_{l-1}],$$

with $c_i = \min\{a_i, b_i\}$. We denote this corner sum $\mathbf{a} \oplus \mathbf{b}$.

Example 6.3.2. The corner sum of two jump sequences can be thought of as the jump sequence obtained by superimposing the Betti diagrams of two edge ideals I_G and $I_{G'}$ on top of one another. In this case, the Betti table of I_G lies above the dashed line, and the Betti table of I_G' lies above the solid line. Betti numbers of solely I_G are indicated by the $\star$ and Betti numbers of solely $I_{G'}$ are indicated by $\circ$. The jump sequence of I_G is $[4; 2, 11, 20]$ and the jump sequence of $I_{G'}$ is $[4; 3, 8, 13]$, with the corner sum of these two jump sequences given by $[4; 2, 8, 14]$.

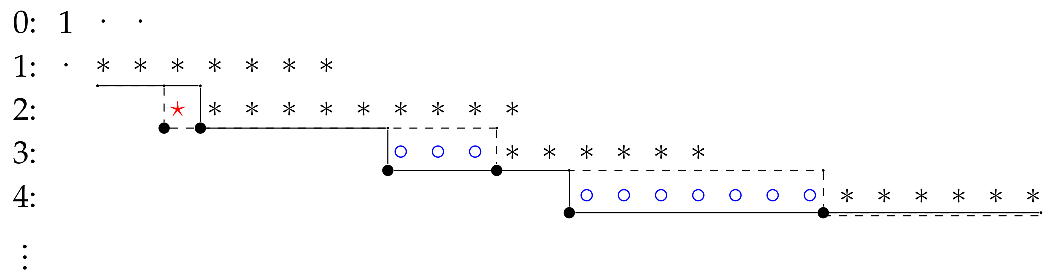


Figure 6.2: Corner Sum Example

We use this corner sum to describe possible jump sequences as follows:

Proposition 6.3.3. Given two Betti diagrams of edge ideals I_G, I_H, with jump sequences $\text{Jump}(I_G)$ and $\text{Jump}(I_G)$ respectively, we have a graph K such that I_K has jump sequence $\text{Jump}(I_K) = \text{Jump}(I_G) \oplus \text{Jump}(I_H)$.

Proof of Proposition 6.3.3. Let $[k; a_1, ..., a_{k-1}]$ and $[l; b_1, ..., b_{l-1}]$ be the jump sequences of I_G and I_H respectively. From Proposition 6.2.2, we can see that $\beta_{i,i+s}(I_K)$ will be nonzero precisely when one or the other of $\beta_{k,k+s}(I_G)$ or $\beta_{k,k+s}(I_K)$ is nonzero, for some $k \leq i$.

In terms of the Betti numbers on the right edge of the Betti table, we see then that the minimal nonzero Betti number in each row should be in position c_i, where

$$c_i = \min\{a_i, b_i\}.$$

This completes our proof, and we can see that the lower edge of the Betti table of I_K is obtained by superimposing the lower edges of the Betti tables of I_G and I_H. This gives us an edge ideal with jump sequence $[l; c_1, c_2, ..., c_{k-1}, b_k, ..., b_l]$ as described. $\square$

As a result, the shapes of Betti diagrams of edge ideals can be imbued with a monoid structure. In Example 6.3.4, we use this to construct a relative jump sequence which is not weakly increasing.

Proposition 6.3.4. Let $4 \leq n_1 \leq n_2 \leq \cdots \leq n_r$ be a set of integers, and form the graphs $G_1 = C^c_{n_1}, G_2 = C^c_{n_2}, ..., G_r = C^c_{n_r}$ on vertex sets $V_i = \{v_{i,j} : 1 \leq j \leq n_i\}$ for $1 \leq i \leq r$. Then the graph on vertex set $V = \cup V_i$ of $G = \cup G_i$ has an edge ideal with reg$(I_G) = 2r+1$, with relative jump sequence

$$\mathbf{r} = [2r; \overbrace{1, 1, ..., 1}^{r-1}, n_1 - 3, n_2 - 3, ..., n_r - 3].$$

Proof. This follows from a quick note on the description of Δ_G in terms of the Δ_{G_i}. As in general, the Stanley-Reisner complex Δ_I of a monomial ideal $I = J + K$, where J and K are monomial ideals on disjoint sets of variables, satisfies

$$\Delta_I \cong \Delta_J * \Delta_K,$$

i.e. Δ_I is the join of the two subcomplexes Δ_J and Δ_K. In particular, we have

$$\Delta_G = \Delta_{G_1} * \Delta_{G_2} * \cdots * \Delta_{G_r}.$$

As a result, our Betti diagram B of the edge ideal of G can be written as the products as matrices of the Betti diagrams B_i of the G_i, with $B = B_1 B_2 \cdots B_r$. The regularity count and the jump sequence calculation follow from an immediate linear algebra computation. □

We use edge ideals of this form, in conjunction with Proposition 6.3.3, to construct an example of an edge ideal whose jump sequence is not weakly increasing.

Proposition 6.3.5. There exists a graph G with relative jump sequence relJump$(I_G) = [k; r_1, ..., r_{k-1}]$ such that $r_i \geq r_{i+1}$ for some i.

Proof. Let $G_1 = C_5^c \cup (C_5^c)' \cup (C_5^c)''$ and $G_2 = C_4^c \cup C_6^c \cup (C_6^c)'$ be two graphs, with G_1 the union of three 5-anticycle graphs and G_2 the union of one 4-anticycle and two 6-anticycles, all viewed as graphs on disjoint sets of vertices. Using Proposition 6.3.4, we have the relative jump sequences of G_1 and G_2 are respectively relJump$(I_{G_1}) = [6; 1, 1, 2, 2, 2]$ and relJump$(I_{G_2}) = [6; 1, 1, 1, 3, 3]$. This gives us jump sequences

$$\text{Jump}(I_{G_1}) = [6; 1, 2, 4, 6, 8] \quad \text{and} \quad \text{Jump}(I_{G_2}) = [6; 1, 2, 3, 6, 9].$$

So using Proposition 6.3.3, we have a graph G which has jump sequence

$$\text{Jump}(G) = \text{Jump}(I_{G_1}) \oplus \text{Jump}(I_{G_2}) = [6; 1, 2, 3, 6, 8],$$

which gives us a relative jump sequence relJump$(I_G) = [6; 1, 1, 1, 3, 2]$. □

Remark 6.3.6. Another interesting question to ask is what additional necessary conditions are required for G to guarantee an increasing relative jump sequences. Alternately, it would be of combinatorial interest to find classes of complexes where these relative jump sequences are not weakly increasing and with Δ connected [excluding trivial cases like coning over a vertex to connect these two tori, etc.]

6.4 Classes of Graphs with indMatch(G) $= 1$ and High Regularity

Theorem 6.4.1. Fix $n \geq 5$. Let H be the graph on vertex set $\{x_1, ..., x_n, y_1, ..., y_n, z_1, z_2\}$, with edges of the following forms:

1. $\{x_i z_1 : 1 \leq i \leq n\}$

2. $\{y_i z_2 : 1 \leq i \leq n\}$

3. $\{x_i y_i : 1 \leq i \leq n\}$

4. $\{x_i y_{i+1} : 1 \leq i \leq n-1\}$

5. $\{y_1 x_n\}$

Then $G = H^c$ has a Gorenstein edge ideal I_G and a shellable Stanley-Reisner complex $\Delta_G = \widehat{H^c}$. This ideal has jump sequence $[3; 2, 2n-2]$.

Proof. This is clear from an examination of the clique closure of H. Unfolding

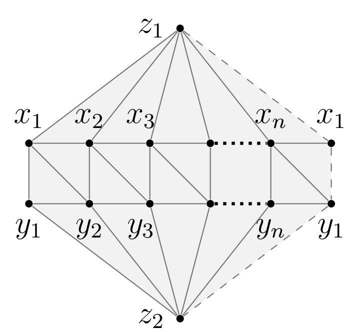


Figure 6.3: Gorenstein edge ideal I_G with jump sequence $[3; 2, 2n-2]$.

the complex, we can see that it is homotopic to a 2-sphere, and by direct examination we note that it is both clique closed and 4-cycle free. As these can be realized as convex triangulations of S^2, ideals of this form are Gorenstein. The smallest induced cycles are of length 5 [for example, $\{x_1y_1, x_1x_2, x_2y_3, y_3z_2, y_1z_2\}$,] and there are 2n+2 total vertices in G, so we have the desired jump sequence. In the case of $n = 5$, we obtain Example 5.0.10, which was the icosahedron. $\square$

All examples edge ideals with jump sequences $[3; a_1, a_2]$ considered so far have had $a_1 = 1$ or $a_1 = 2$. We present a (non-Cohen-Macaulay) example of a complex with $a_1 = 3$ and $a_2 = nm - 4$ for any $n, m \geq 6$.

Example 6.4.2. Let H be a graph on vertex set $\{x_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\}$, with edges of the following forms:

1. $\{x_{i,j}x_{i,j+1} : 1 \leq i \leq n, 1 \leq j \leq m-1\}$
2. $\{x_{i,1}x_{i,n}, 1 \leq i \leq n\}$
3. $\{x_{i,j}x_{i+1,j} : 1 \leq i \leq n-1, 1 \leq j \leq m\}$
4. $\{x_{1,j}x_{n,j}, 1 \leq j \leq m\}$
5. $\{x_{i,j}x_{i+1,j+1} : 1 \leq i \leq n, 1 \leq j \leq m\}$
6. $\{x_{i,1}x_{i+1,m} : 1 \leq i \leq n-1\}$
7. $\{x_{1,j}x_{n,j+1} : 1 \leq j \leq m-1\}$
8. $\{x_{1,1}x_{n,n}\}$

Then $G = H^c$ will have an edge ideal with jump sequence $[3; 3, nm - 4]$.

This is clear from an examination of H, which forms the 1-skeleton of Δ_G. The clique closure of this 1-skeleton is a torus on nm vertices, with smallest induced cycles of length 6. No induced proper subcomplex has nonzero second homology, so we must have jump sequence $[3; 3, nm - 4]$.

Remark 6.4.3. Each of these classes of graphs with indMatch(G) $= 1$ and regularity 3,4, or 5 give rise to edge ideals with indMatch(G′) $= k$ and regularity reg$(I_{G'}) = 2k + 1$, reg$(I_{G'}) = 3k + 1$, and reg$(I_{G'}) = 4k + 1$, respectively. Given a graph G with regularity r, taking G' to be k disjoint copies of the graph on different sets of variables gives a Stanley-Reisner complex:

$$\Delta_{G'} = \Delta_G * \Delta_G * \cdots \Delta_G,$$

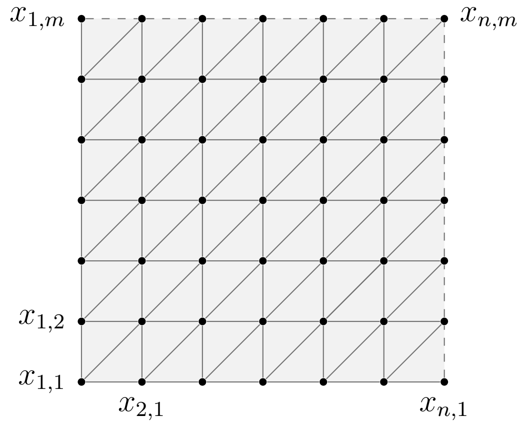


Figure 6.4: Edge ideal I_G of the complement of a triangulation of a torus with jump sequence $[3; 3, nm - 4]$.

via combinatorial joins of the faces in Δ_G. Via the Künneth formula, we see we have nonzero homology in the desired degrees, giving us the desired regularity calculation.

6.5 Modified Barycentric Subdivision

A more general way of constructing graphs with a C_4-free complement is desirable. Given any triangulation of a 2-sphere Δ, there is a way of retriangulating the sphere to produce a new complex $\mathrm{sd}_4(\Delta)$ which is the Stanley-Reisner complex of an edge ideal I_G with a C_4-free 1-skeleton.

Definition 6.5.1. Let Δ be a pure dimensional simplicial complex whose facets are all of dimension 2. Then $\mathrm{sd}_4(\Delta)$ is the simplicial complex obtained by replacing each facet with the following complex:

Proposition 6.5.2. Let Δ be a simplicial triangulation of a 2-sphere. Then the Stanley-Reisner ideal of $\mathrm{sd}_4(\Delta_G)$ is generated in degree 2. Viewing this ideal as

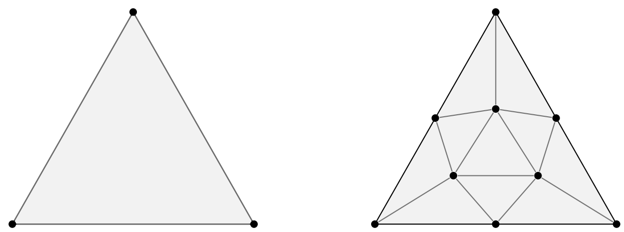


Figure 6.5: New Triangulation Δ of a Sphere with indMatch(G) $= 1$ for the complement graph $G^c = \Delta_1$.

an edge ideal of a graph G, we have that G^c is C_4-free.

Proof. As there are no induced 4-cycles inside an individual face $\mathrm{sd}_4(\sigma)$, we may consider how facets σ, σ' intersect after this subdivision. As we assumed that Δ was a simplicial triangulation of a sphere, any two facets share at most one edge. Along this edge, the only possible induced cycle is of length 6. As every vertex must be in at least 3 facets σ, σ', and σ'', we also note that every cycle obtained as the link of a vertex v must be of length at least 6. Performing all of these checks locally in the triangulation of Δ, we see that the 1-skeleton of $\mathrm{sd}_4(\Delta)$ must be C_4-free.

We have that $\mathrm{sd}_4(\Delta)$ is generated in degree 2 by noting that no boundaries of a 3-simplex can occur, so $\mathrm{sd}_4(\Delta)$ must be clique closed. As all clique complexes have degree 2 generated Stanley-Reisner ideals, the proof is complete. $\square$

Note that this definition did not require that Δ be a clique-closed simplicial triangulation, only a simplicial triangulation. This provides a way of constructing an infinite family of C_4 free edge ideals from a large family of simplicial complexes.

6.6 Conclusions and Future Work

A better understanding of the possible shapes of Betti diagrams of edge ideals is desirable. While a complete classification of the jump sequences Jump(I_G) of edge ideals seems somewhat unrealistic, partial solutions are still of interest in simplicial topology. For example, sharp conditions for a jump sequence of length 2 to exist translate into constraints on the structure of flag triangulations of spheres – an area of general combinatorial interest.

Questions 6.6.1. Additional questions about the Betti numbers and Stanley-Reisner complexes Δ_G of edge ideals include:

1. Can sharp conditions be given on possible jump sequences $[k; a_1, a_2, ..., a_{k-1}]$? Can sharp conditions even be given on jump sequences $[k; a_1, a_2]$, i.e. for graphs with reg$(I_G) = 4$?
2. Are the Betti diagrams of I_G strand connected, i.e. if $\beta_{i,j}(I_G)$ and $\beta_{i+k,j+k}(I_G)$ are both nonzero, are $\beta_{i+k',j+k'}(I_G) \neq 0$ for all $0 \leq k' \leq k$? [This is known for the linear strand, but not even for the first nonlinear strand.]
3. Do there exist graphs with regularity higher than indMatch(G) but lower than the co-chordal clutter size of G?
4. Do there exist graphs with indMatch(G) $= 1$ and reg$(I_G) = 6$?
5. Do there exist graphs with indMatch(G) $= k$ and reg$(I_G) \geq 4k + 2$?

CHAPTER 7

STABILIZATION OF BETTI TABLES

7.0.1 Asymptotics of Regularity of I^d

For an ideal $I \subseteq R = \Bbbk[x_1, ..., x_n]$, much work has been done on showing that the Castelnuovo-Mumford regularity of I^d is a linear function in terms of d for high powers. The following theorem is a result of Cutkosky, Herzog and Trung:

Theorem 7.0.2 (Theorem 1.1 in [3]). Let I be an arbitrary homogeneous ideal. Let $r(I)$ denote the maximum degree of the homogeneous generators of I. The following hold:

(i) There is a number e such that reg$(I^d) \leq d \cdot r(I) + e$ for all $d \geq 1$.

(ii) reg(I^d) is a linear function for all d large enough.

They provide criterion for estimating this e in the case of an equigenerated ideal I, i.e. an ideal generated by homogeneous generators of the same degree. This result generalizes an earlier bound by Swanson giving the existence of k such that

$$\text{reg}(I^d) \leq kd$$

for homogeneous ideals in [33].

Let $I \subseteq R = \Bbbk[x_1, ..., x_n]$ be an ideal. The *graded Betti numbers* of a homogeneous ideal I are given by $\beta_{i,j}(I) = \dim_{\Bbbk} \text{Tor}_i(\Bbbk, I)_j$. The graded Betti numbers also correspond to the ranks of the free modules in a minimal free resolution of I. We organize this data into the *Betti table of I* (in the style of Macaulay 2) displaying $\beta_{i,i+j}(R/I)$ in the i^{th} column and j^{th} row, as seen in Example 7.0.5.

We recall the definition of a singly graded equigenerated ideal.

Definition 7.0.3. We say that an ideal $I = (f_0, f_1, ..., f_k) \subset R = \Bbbk[x_1, ..., x_N]$ is *equigenerated in degree* r if $\deg(f_i) = r$ for all f_i.

Using techniques similar to those in [3], and [1], we produce here a sharper result on the asymptotics of Betti tables of powers I^d.

Theorem 7.0.4 (Theorem 7.3.1)**.** Let $I = (f_0, f_1, ..., f_k) \subseteq \Bbbk[x_1, ..., x_n] = R$ be an equigenerated ideal of degree r. Then there exists a D such that for all $d > D$, we have

$$\beta_{i,j+rd}(I^d) \neq 0 \Longleftrightarrow \beta_{i,j+rD}(I^D) \neq 0.$$

This gives us that the shape of the Betti tables of powers of an ideal I is eventually fixed, translated down by the degree r of the ideal.

Example 7.0.5. Let $I = (x_3x_4x_5, x_1x_6x_7, x_3x_6x_8, x_1x_5x_9, x_2x_8x_9) \subseteq \Bbbk[x_1, ..., x_9]$. We consider the Betti diagrams of the resolutions of the first few powers I^d of I. The diagrams have been shifted to only show nonzero Betti numbers in the resolution of I^d.

I

-	1	2	3	4	5
total:	5	10	9	3	·
2:	5	·	·	·	·
3:	·	6	·	·	·
4:	·	4	9	3	·

I^2

-	1	2	3	4	5
total:	15	41	39	12	·
5:	15	·	·	·	·
6:	·	33	12	·	·
7:	·	8	27	12	·

I^3

-	1	2	3	4	5
total:	35	117	121	39	1
8:	35	·	·	·	·
9:	·	105	67	9	·
10:	·	12	54	30	1

I^4

-	1	2	3	4	5
total:	70	271	302	105	5
11:	70	·	·	·	·
12:	·	255	212	45	·
13:	·	16	90	60	5

I^5

-	1	2	3	4	5
total:	126	545	645	240	15
14:	126	·	·	·	·
15:	·	525	510	135	·
16:	·	20	135	105	15

I^6

-	1	2	3	4	5
total:	210	990	1229	483	35
17:	210	·	·	·	·
18:	·	996	1040	315	·
19:	·	24	189	168	35

We can see the stabilized shape of the powers of I^d will be:

I^d

-	1	2	3	4	5
total:	$*$	$*$	$*$	$*$	$*$
3d-1:	$*$	·	·	·	·
3d:	·	$*$	$*$	$*$	·
3d+1:	·	$*$	$*$	$*$	$*$

Unfortunately, Theorem 7.3.1 does not guarantee that powers of our ideals I^d will have linear resolutions if the resolution of I^l is linear for some l with $d > l$. As a counterexample, we have the following example (due to Sturmfels):

Example 7.0.6 (Theorem 1.1 in [32])**.** Set

$$I = (def, cef, cdf, cde, bef, bcd, acf, ade) \subseteq \Bbbk[a, b, c, d, e, f].$$

The ideal I has linear resolution and linear quotients with respect to the ordering given above, but I^2 fails to be linear. We include the Betti tables of I and I^2 here.

I	-	0	1	2	3
	total:	1	8	11	4
	0:	1	·	·	·
	1:	·	·	·	·
	2:	·	8	11	4

I^2	-	0	1	2	3	4	5	6
	total:	1	36	85	79	38	10	1
	0:	1	·	·	·	·	·	·
	1:	·	·	·	·	·	·	·
	2:	·	·	·	·	·	·	·
	3:	·	·	·	·	·	·	·
	4:	·	·	·	·	·	·	·
	5:	·	36	84	75	32	6	·
	6:	·	·	1	4	6	4	1

More generally, Conca provided a class of ideals I_k which have linear quotients (and hence, linear resolutions) until the k^{th} power, then have nonlinear resolutions for all powers higher than k [2]. This implies that for an ideal I, the shapes of Betti tables of $I, I^2, \ldots, I^d$ and I^{d+1} need not satisfy any chain of inclusions, though they eventually stabilize for some I^D.

We also provide an upper bound for the Betti numbers of powers of an equigenerated ideal I in terms of the Betti numbers of the Rees ideal of I as follows.

Theorem 7.0.7 (Theorem 7.2.1)**.** Let $I = (f_0, f_1, ..., f_k) \subseteq R = \Bbbk[x_1, ..., x_N]$ with f_i homogeneous of degree r. Let $\mathcal{R}(I)$ be the Rees algebra of I in ring $S = \Bbbk[x_1, ..., x_N, w_0, ..., w_k]$ with bigrading $\deg(x_i) = (1, 0)$ and $\deg(w_i) = (0, 1)$. Then

$$\beta_{i,j+rd}(I^d) \leq \sum_{m=0}^{d} \binom{d+k-m}{d-m} \beta_{i,(j,m)}(\mathcal{R}(I))$$

holds for all i, j, d.

The proof follows from a careful examination of the restriction of a minimal resolution of $\mathcal{R}(I)$ to bidegrees $(*, d)$. We give the smallest D for which this stabilization occurs a name:

Definition 7.0.8 (Definition 7.4.1)**.** Let I be a homogeneous equigenerated ideal in polynomial ring R. Let the *stabilization index Stab*(I) *of* I be the smallest such D such that for all $d \geq D$,

$$\beta_{i,j+rd}(I^d) \neq 0 \Longleftrightarrow \beta_{i,j+rD}(I^D) \neq 0.$$

Finding Stab(I) in directly in terms algebraic properties of I remains open, although a conjecture for edge ideals will be given in Section 7.4. Areas of future research include producing explicit Stab(I) for other classes of ideals or providing sharper bounds for Stab(I) than those included here.

7.1 Rees Algebras of Equigenerated Ideals

Taking a resolution (with an appropriately chosen bigrading) of L gives resolutions of all powers of L, and can be used to bound or explicitly compute Betti numbers $\beta_{i,j}(I^n)$ for all n.

We will assume throughout this Chapter that $I = (f_0, f_1, ..., f_k)$ is an equigenerated ideal of degree r in $R = \mathbb{k}[x_1, ..., x_N]$. Notationally, we set $\mathcal{R}(I) = S/L$ with L the Rees ideal of I and $S = \mathbb{k}[x_1, ..., x_N, w_0, w_1, ..., w_k]$.

We bigrade $\mathcal{R}(I)$ by $\deg(x_i) = (1, 0)$ and $\deg(w_i) = (0, 1)$ and take the minimal graded free resolution of $\mathcal{R}(I)$ with respect to this grading.

$$\mathcal{F}: \quad \mathcal{R}(I) \leftarrow S \leftarrow \bigoplus_{(j,m)} S(-j, -m)^{\beta_{1,(j,m)}} \leftarrow \cdots \leftarrow \bigoplus_{(j,m)} S(-j, -m)^{\beta_{p,(j,m)}} \leftarrow 0.$$

Restricting to the strand $(*, d)$, we obtain a (possibly nonminimal) resolution of

I^d:

$$\mathcal{F}_d: \quad I^d \leftarrow S_{(*,d)} \leftarrow \bigoplus_{(j,m)} S(-j,-m)^{\beta_{1,(j,m)}}_{(*,d)} \leftarrow \cdots \leftarrow \bigoplus_{(j,m)} S(-j,m)^{\beta_{p,(j,m)}}_{(*,d)} \leftarrow 0. \tag{7.1}$$

Tensoring this resolution with $\Bbbk$ and taking the homology of the maps computes us $\dim \operatorname{Tor}_i^R(\Bbbk, I^d)_{j+rd} = \beta_{i,j+rd}(I^d)$. This shift in the indices of $\beta_{i,j+rd}(I^d)$ accounts for the shift in grading to agree with that of R while viewing I^d as an R module.

Alternately, we could have first tensored with S/M for $M = (x_1, ..., x_N)$, taken homology of our maps, then restricted in degrees. This will give us modules $\operatorname{Tor}_i^S(S/M, \mathcal{R}(I))_j$, and as these two actions commute, we have that

$$\begin{aligned} \operatorname{Tor}_i^S(S/M, \mathcal{R}(I))_{(j,d)} &= \operatorname{Tor}_i^S(S/M, I^d)_j \\ &= \operatorname{Tor}_i^R(\Bbbk, I^d)_{j+rd}. \end{aligned}$$

Hence we have that all Betti numbers of higher powers can be written in terms of the dimensions of the bigraded modules $\operatorname{Tor}_i^S(S/M, \mathcal{R}(I))$, given by

$$\beta_{i,j+rd}(I^d) = \dim \operatorname{Tor}_i^S(S/M, \mathcal{R}(I))_{(j,d)}.$$

7.2 Bounds on Betti Numbers of Powers of Ideals

We resolve the Rees algebra $\mathcal{R}(I)$ and restrict to fixed w-degree strands to produce explicit bounds on the Betti numbers of I^d.

Theorem 7.2.1. Let $I = (f_0, f_1, ..., f_k) \subseteq R = \Bbbk[x_1, ..., x_N]$ with all f_i homogeneous of degree r. Let $\mathcal{R}(I)$ be the Rees algebra of I in ring $S =$

$\Bbbk[x_1, ..., x_N, w_0, ..., w_k]$ with bigrading $\deg(x_i) = (1, 0)$ and $\deg(w_i) = (0, 1)$. Then

$$\beta_{i,j+dr}(I^d) \le \sum_{m=0}^{d} \binom{d+k-m}{d-m} \beta_{i,(j,m)}(\mathcal{R}(I))$$

holds for all i, j, d.

Proof. We take a minimal free resolution of $\mathcal{R}(I)$ and consider the degree restricted strand used in Section 7.1:

$$\mathcal{F}_d: \quad I^d \leftarrow S_{(*,d)} \leftarrow \bigoplus_{(j,m)} S(-j,-m)_{(*,d)}^{\beta_{1,(j,m)}} \leftarrow \cdots \leftarrow \bigoplus_{(j,m)} S(-j,-m)_{(*,d)}^{\beta_{p,(j,m)}} \leftarrow 0.$$

Let $T = \Bbbk[w_0, w_1, ..., w_k]$ be the polynomial ring in the w_i-variables. Then we can rewrite our bigraded pieces $S(-j,-m) = R(-j) \otimes T(-m)$. Then in a fixed strand $(*, d)$, we have:

$$\mathcal{F}_d: \quad I^d \leftarrow R \otimes T_d \leftarrow \bigoplus_{(j,m)} R(-j) \otimes T(-m)_d^{\beta_{1,(j,d)}} \leftarrow \cdots \leftarrow \bigoplus_{(j,m)} R(-j) \otimes T(-m)_d^{\beta_{p,(j,d)}} \leftarrow 0.$$

It remains to count the dimension over R of the i^{th} module

$$F_i = \bigoplus_{(j,m)} R(-j) \otimes T(m)_d^{\beta_{i,(j,m)}(\mathcal{R}(I))}$$

in a fixed degree $j + rd$ of the resolution. Finally, the dimension of $T(-m)_d$ is the number of degree $d - m$ monomials in a polynomial ring in $k + 1$ variables, or

$$\binom{d+k-m}{k}.$$

So we have that

$$\beta_{i,j+rd}(I^d) \le \sum_{m=0}^{d} \binom{d+k-m}{k} \beta_{i,(j,m)}(\mathcal{R}(I)),$$

proving the theorem. $\square$

This immediately shows that the Betti diagram of I^d sits inside an (appropriately degree shifted) table coming from the Betti diagram of the resolution of $\mathcal{R}(I)$. This implies that the number of nonzero graded Betti numbers of I^d is bounded independent of the power d. We refine this rough bound in the following section.

7.3 Betti Diagrams of Powers of Stanley-Reisner Ideals I_Δ

We are now ready to prove the main theorem:

Theorem 7.3.1 (Betti Tables of Powers of Equigenerated Ideals).
Let $I = (f_0, f_1, ..., f_k) \subseteq \Bbbk[x_1, ..., x_N] = R$ be an equigenerated ideal of degree r. Then there exists a D such that for all $d > D$, we have

$$\beta_{i,j+rd}(I^d) \neq 0 \iff \beta_{i,j+rD}(I^D) \neq 0.$$

Proof of Theorem 7.3.1. From the calculation in Section 7.1, we have that

$$\beta_{i,j+rd}(I^d) = \dim \mathrm{Tor}_i^S(S/M, \mathcal{R}(I))_{(j,d)}.$$

The $\mathrm{Tor}_i(S/M, \mathcal{R}(I))$ are finitely generated bigraded S-modules. We decompose them into bigraded components in the following way.

Let $M_i := \mathrm{Tor}_i(S/M, \mathcal{R}(I))$ and $M_{ij} := (M_i)_{(j,*)}$. The M_{ij} are finitely generated graded T-modules, where $T = \Bbbk[w_0, w_1, ..., w_k]$ is the polynomial ring in the w_i-variables. So each M_{ij} has a Hilbert polynomial such that

$$P_{ij}(d) := P_{M_{ij}}(d) = \dim(M_i)_{(j,d)}$$

for all $d \geq d_{ij}$, with d_{ij} the regularity of M_{ij} as a T-module. Hence, for all $d \geq d_{ij}$ and $P_{M_{ij}}$ not identically zero, we have $\beta_{i,j+dr}(I^d) = \dim(M_i)_{(j,d)} = P_{M_{i,j}}(d) > 0.$

Note that $D = \max_{i,j}\{D_{ij}\}$ will be an upper bound for Stab(I), providing such a maximum exists.

Lemma 7.3.2. There are only finitely many nonzero M_{ij}.

Proof of Lemma 7.3.2. That only finitely many M_j are nonzero follows from

$$\beta_{i,j+rd}(I^d) = \dim \mathrm{Tor}_i^S(S/M, \mathcal{R}(I))_{(j,d)}.$$

As the projective dimension of all powers I^d is bounded by N the number of variables in our original ring, $\mathrm{Tor}_i^S(S/M, \mathcal{R}(I)) = 0$ for all $i > N$.

We now consider a fixed M_i. Theorem 7.2.1 gave a bound on the Betti numbers of I^d depending on the Betti numbers of $\mathcal{R}(I)$,

$$\beta_{i,j+rd}(I^d) \leq \sum_{m=0}^{d} \binom{d+k-m}{d-m} \beta_{i,(j,m)}(\mathcal{R}(I)).$$

As for a fixed i, the number of nonzero Betti numbers of $\mathcal{R}(I)$ must be finite, there can be only finitely many j such that $\beta_{i,(j,m)}(\mathcal{R}(I)) \neq 0$. This implies that for j outside of this set, $\beta_{i,j+rd}(I^d) \leq \sum_{m=0}^{d} 0$ for all d, which implies $\beta_{i,j+rd}(I^d) = 0$. So $M_{ij} = 0$ except for a finite number of cases.

This completes the proof of the lemma. $\square$

By Lemma 7.3.2, we have that the maximum

$$D = \max_{i,j}\left\{D_{ij}\right\}$$

exists. Hence, we have that

$$\dim \mathrm{Tor}_i(S/M, \mathcal{R}(I)))_{(*,d)} = P_{M_i}(d)$$

is a polynomial function for all $d > D$. We note that for all $d > D$,

$$\beta_{i,j+dr}(I^d) = \dim(M_i)_{(j,d)} = P_{M_{i,j}}(d) > 0$$

if and only if

$$\beta_{i,j+Dr}(I^D) = \dim(M_i)_{(j,D)} = P_{M_{i,j}}(D) > 0,$$

completing the proof. $\square$

The techniques used throughout the proof of Theorem 7.3.1 were similar to those seen in [1], [3], and [33], but extend their results to a classification of all possible nonzero graded Betti numbers of powers of an equigenerated ideal I.

7.4 Stabilization Index of I

The bound D produced in Theorem 7.3.1 is not sharp, and finding the smallest such D, which we will call the *stabilization index of I* Stab(I), in terms of combinatorial data of I is a subject of future research.

Definition 7.4.1. Let I be a homogeneous ideal equigenerated in degree r in polynomial ring R. Let Stab(I) be the smallest such D such that for all $d \geq D$,

$$\beta_{i,j+rd}(I^d) \neq 0 \iff \beta_{i,j+rD}(I^D) \neq 0.$$

While this is unknown in general, we conjecture here a formula for Stab(I_G) for edge ideal I_G.

Conjecture 7.4.2. Let $I_G = (m_0, m_1, ..., m_k) \subseteq \Bbbk[x_1, ..., x_N]$ be a square-free monomial ideal with $\cup_i \mathbf{supp}(m_i) = \{x_1, ..., x_N\}$. Then

$$\text{Stab}(I_G) = \min\{n : \text{there exists an } \mathbf{m} \in I_G^n \text{ such that } x_i^2 | \mathbf{m} \text{ for all i.}\}$$

This seems to be related to the Stanley-Reisner complexes of polarization of the powers of the edge ideal, but a clear proof that the Betti diagrams stabilize from the existence of such a generator is still unknown. Finding a formula for Stab(I) of other monomial ideals I remains open.

7.4.1 Areas of Future Research

We would like to answer the following questions in subsequent work on these stabilization indices:

1. Do formulas for Stab(I) exist for squarefree monomial ideals? Do they relate to the dimensions of the facet complex or the Stanley-Reisner complex?
2. Does Stab(I_Δ) have a topological interpretation in terms of $\Delta_{\text{pol}(I^n)}$, the Stanley-Reisner complex of the polarization of I^n?
3. Does there exist a class of ideals for which the D produced in Theorem 7.3.1 is the sharp bound, i.e. $D = $ Stab(I)?

Aside from the stabilization index, the shapes of chain of Betti tables leading up to the stabilized Betti table appear fairly interesting. Generally, the shapes of Betti tables of powers of homogeneous equigenerated ideals seem to be unimodal, in the following sense:

Conjecture 7.4.3. Let $I \subseteq R$ be an equigenerated homogeneous ideal generated in degree r. Then for each pair of indices (i, j) there exist $1 \leq D_1 \leq D_2 \leq \infty$ such that for all d with $D_1 \leq d \leq D_2$,

$$\beta_{i,j+dr}(I^d) \neq 0$$

and for all $d < D_1$ or $D_2 < d$,

$$\beta_{i,j+dr}(I^d) = 0.$$

Proving this conjecture would require a better understanding of the modules M_{ij} described above. These M_{ij} seem to carry interesting structure, and investigating the connections between M_{ij} and the geometry of the ideal I and its Rees algebra $\mathcal{R}(I)$ is another area of future interest.

CHAPTER 8

LINEAR QUOTIENTS ORDERING OF SQUARE OF ANTICYCLE

This chapter is joint work with A. Hoefel.

Let G be a simple graph on n vertices, and $I(G)$ its edge ideal, i.e., a squarefree monomial ideal in $R = \Bbbk[x_1, \ldots, x_n]$ with monomial generators $x_i x_j$ corresponding to each edge $\{i, j\} \in G$. Such ideals have been extensively studied in such papers as [15], [16], [28], [37], and more recently, [26]. A goal of much recent research has been to classify behavior of the resolutions of such ideals $I(G)$ and that of their powers in terms of combinatorial data of G. We provide here an explicit proof that the second power of the edge ideal of the anticycle has not just a linear resolution, but also linear quotients.

In the course the proof, we additionally demonstrate that all powers $I(P_n^c)^k$ of the edge ideal of the antipath have linear quotients.

Definition 8.0.4. Let G be a simple graph on n vertices. Then the *edge ideal of* G is the squarefree monomial ideal $I(G)$ given by

$$I(G) = (x_i x_j : \{i, j\} \in G).$$

We say that a graph G has property P if its edge ideal $I(G)$ has such a property; e.g., G is Gorenstein if $I(G)$ is Gorenstein, G is linear if $I(G)$ has a linear resolution, etc. In particular, we will say a graph G has linear quotients if its edge ideal $I(G)$ has linear quotients:

Definition 8.0.5. Let I be a homogeneous ideal. We say that *I has linear quotients* if there exists some ordering of the generators of $I = (m_1, m_2, \ldots, m_r)$ such that for all $i > 1$,

$$((m_1, \ldots, m_{i-1}) : (m_i)) = (x_{k_1}, \ldots, x_{k_s})$$

for some variables $x_{k_1}, \ldots, x_{k_s}$. We say that such an ordering $(m_1, m_2, \ldots, m_r)$ is a *linear quotients ordering of* I.

For two monomials m and m' we define $m' : m$ to be the monomial $\frac{m'}{\gcd(m,m')}$. Given monomials $m_1, \ldots, m_i$, the colon ideal $(m_1, \ldots, m_{i-1}) : (m_i)$ can be computed as

$$(m_1, \ldots, m_{i-1}) : (m_i) = (m_1 : m_i, \ldots, m_{i-1} : m_i).$$

Thus, in order to show that a monomial ideal $I = (m_1, \ldots, m_r)$ has linear quotients, it suffices to show that for each pair of monomials m_i and m_j with $j < i$ that there exists another monomial m_k with $k < i$ with

$$m_k : m_i = x_l \text{ for some } l \qquad \text{and} \qquad x_l \text{ divides } m_j : m_i.$$

The *graded Betti numbers* of a homogeneous ideal I are given by $\beta_{i,j}(I) = \dim_{\Bbbk} \mathrm{Tor}_i(I, \Bbbk)_j$. The graded Betti numbers also correspond to the ranks of the free modules in a minimal free resolution of I. We say an ideal I which is generated in degree d has a *linear resolution* if $\beta_{i,j}(I) = 0$ for $j \neq i + d$. Ideals with linear quotients also have linear resolutions.

Providing a linear quotients ordering is one technique for proving that an ideal has a linear resolution, often with combinatorial significance in the case of monomial ideals. In the case of squarefree monomial ideal, an ideal I having linear quotients is equivalent to its Alexander dual $I^\vee$ having a shelling order on its facets. For non-squarefree monomial ideals, a linear quotient orderingscan be viewed as giving a shelling order on the Alexander dual of its polarization.

Interest in powers of the anticycle partially draws from a result of Herzog, Hibi and Zheng [18] which states the following:

Theorem 8.0.6 (Herzog, Hibi, Zheng)**.** Let I be a quadratic monomial ideal of the polynomial ring. The following are equivalent:

1. I has a linear resolution,
2. I has linear quotients,
3. I^k has a linear resolution for all $k \geq 1$.

For edge ideals, Fröberg showed that $I(G)$ has a linear resolution if and only if the complement of G is chordal [12].

Conspicuously missing from the above theorem is the statement that all powers of a quadratic monomial ideal I with linear resolution must have linear quotients. In fact, this is not known. There are numerous examples of non-quadratic monomial ideals possessing a linear resolution, or even linear quotients, whose powers do not. In [2], Conca provides a example generated in degree 3 which is not dependent on the characteristic of the field $\mathbb{k}$.

It would be of interest to construct linear quotients of powers of quadratic monomial ideals with the aim of extending Herzog, Hibi and Zheng's theorem. Alternately, as no counterexamples are known, the construction of a quadratic monomial ideal I with a linear resolution but some power k with no linear quotients ordering on the generators of I^k would be of combinatorial interest.

Our work on the second power of the anticycle was also inspired by a second thread of research. Francisco, Hà and Van Tuyl first investigated graphs G where $I(G)^k$ has a linear resolution for each $k \geq 2$.

From Fröberg and Herzog, Hibi and Zheng's results, we see that chordal graphs have this property. More generally, it has been shown by Francisco, Hà and Van Tuyl that if some power of $I(G)$ has a linear resolution, then the complement of G cannot contain any induced four cycles. Their proof was recorded in [29].

Inspired by these results, Peeva and Nevo constructed an example of a graph G with no four cycle in its complement and where $I(G)^2$ does not have a linear resolution. Peeva and Nevo have conjectured that their example works only because $I(G)$ has Castelnuovo-Mumford regularity four and that every successive power of an edge ideal should get strictly closer to a linear resolution. See [29] for a more precise statement.

Nevo has also shown that claw-free graphs with no four cycles in their complements have regularity at most three and their second powers have linear resolutions [28]. Anticycles on more than four vertices meet these criteria and so, it follows that their second powers have linear resolutions. Here we demonstrate that the square of the edge ideal of the anticycle has linear quotients, recovering this result.

8.1 Cycles, Anticycles, and Antipaths

We first describe the edge ideal of the anticycle and partition pairs of its edges into several natural classes. Next, we provide a linear quotients ordering on these classes relative to the previous generators.

The *complement* of a graph G is the graph on the vertices of G containing all edges that are not in G. We use G^c to denote the complement graph.

Definition 8.1.1. Let C_n be the cycle graph on n vertices, i.e. the graph consisting of one cycle of length n on these vertices with no chords. The *anticycle graph* A_n is the complement graph of C_n, i.e., $A_n = C_n^c$.

Definition 8.1.2. The *antipath* P_n^c is the graph on n vertices containing of all edges in the complement of a path P_n of length $n-1$. We depict the antipath in the figure below.

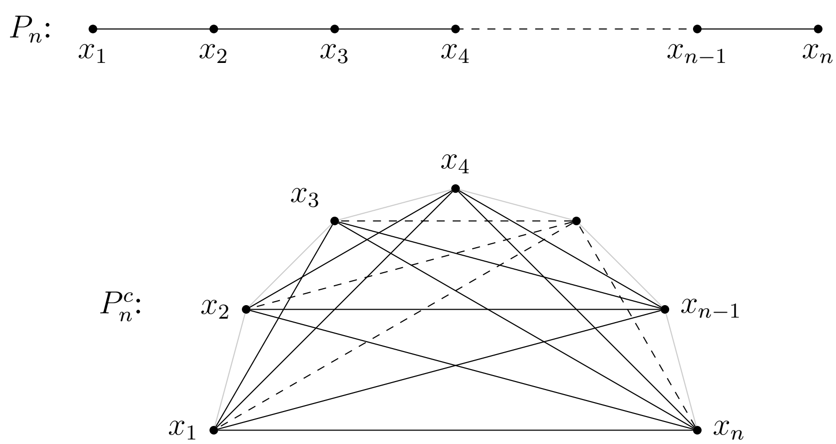


Figure 8.1: H^c a Graph of n-Path, H is Graph of the n-Antipath

Producing a linear quotients ordering for graphs with chordal complements is always possible and all of their powers have linear resolutions, as given in Theorem 3.2 in [18]. However, most naive orderings on the generators of higher powers of $I(G)$ fail to produce linear quotients for G with chordal complements.

Example 8.1.3. Let $R = \mathbb{k}[x_1, \ldots, x_6]$ and let $I = I(A_n)^2$ be the square of the edge ideal of the anticycle on 6 vertices in R. Its generators, written in lex order, are given by:

$$x_1^2x_3^2, x_1^2x_3x_4, x_1^2x_3x_5, x_1^2x_4^2, x_1^2x_4x_5, x_1^2x_5^2, x_1x_2x_3x_4, x_1x_2x_3x_5, x_1x_2x_3x_6,$$
$$x_1x_2x_4^2, x_1x_2x_4x_5, x_1x_2x_4x_6, x_1x_2x_5^2, x_1x_2x_5x_6, x_1x_3^2x_5, x_1x_3^2x_6, x_1x_3x_4x_5,$$
$$x_1x_3x_4x_6, x_1x_3x_5^2, x_1x_3x_5x_6, x_1x_4^2x_6, x_1x_4x_5x_6, x_2^2x_4^2, x_2^2x_4x_5, x_2^2x_4x_6, x_2^2x_5^2,$$
$$x_2^2x_5x_6, x_2^2x_6^2, x_2x_3x_4x_5, x_2x_3x_4x_6, x_2x_3x_5^2, x_2x_3x_5x_6, x_2x_3x_6^2, x_2x_4^2x_6,$$
$$x_2x_4x_5x_6, x_2x_4x_6^2, x_3^2x_5^2, x_3^2x_5x_6, x_3^2x_6^2, x_3x_4x_5x_6, x_3x_4x_6^2, x_4^2x_6^2.$$

This ordering *fails* to be a linear quotients ordering. Let m_i be the i^{th} monomial in the ordering above, and let I_i denote the ideal generated by the first $i-1$ monomials in the ordering. Setting $Q_i = I_i : (m_i)$, we see that

$$\begin{aligned} Q_9 &= (x_1^2x_3^2, x_1^2x_3x_4, x_1x_2x_3x_4, x_1^2x_4^2, x_1x_2x_4^2, x_2^2x_4^2, x_1^2x_3x_5, x_1x_2x_3x_5) : (x_1x_2x_3x_6) \\ &= (x_4, x_5, x_1x_3) \end{aligned}$$

is not generated by variables, hence the lex ordering fails to give us linear quotients. Similarly, with reverse lex, we have the following ordered generating set:

$$x_1^2x_3^2, x_1^2x_3x_4, x_1x_2x_3x_4, x_1^2x_4^2, x_1x_2x_4^2, x_2^2x_4^2, x_1^2x_3x_5, x_1x_2x_3x_5, x_1x_3^2x_5,$$
$$x_1^2x_4x_5, x_1x_2x_4x_5, x_2^2x_4x_5, x_1x_3x_4x_5, x_2x_3x_4x_5, x_1^2x_5^2, x_1x_2x_5^2, x_2^2x_5^2, x_1x_3x_5^2,$$
$$x_2x_3x_5^2, x_3^2x_5^2, x_1x_2x_3x_6, x_1x_3^2x_6, x_1x_2x_4x_6, x_2^2x_4x_6, x_1x_3x_4x_6, x_2x_3x_4x_6,$$
$$x_1x_4^2x_6, x_2x_4^2x_6, x_1x_2x_5x_6, x_2^2x_5x_6, x_1x_3x_5x_6, x_2x_3x_5x_6, x_3^2x_5x_6, x_1x_4x_5x_6,$$
$$x_2x_4x_5x_6, x_3x_4x_5x_6, x_2^2x_6^2, x_2x_3x_6^2, x_3^2x_6^2, x_2x_4x_6^2, x_3x_4x_6^2, x_4^2x_6^2.$$

This fails to have linear quotients at $Q_{21} = I_{21} : (x_1x_2x_3x_6) = (x_4, x_5, x_1x_3)$. Using a monomial ordering on the generators of I does not appear to ever produce a linear quotients ordering on the generators of $I(A_n)^2$.

This appears to be true more generally – while all higher powers of edge ideals with linear quotients appear to have linear quotients as well, these linear quotients orders almost never arise from a monomial term ordering.

8.2 Antipath Linear Quotients

Throughout this section we will use $H = P_n^c$ to denote the antipath on n vertices. The first stage in our linear quotients ordering is to show that the square of the antipath has linear quotients with respect to the lex order. As the complement of the antipath is a chordal graph, it is known that $I(H)$ has a linear resolution via Fröberg's Theorem [12]. Furthermore, as $I(H)$ has a linear resolution and is generated in degree 2, it is known to have a linear quotient ordering and linear resolutions of all of its powers [18]. However, a linear resolution of its second power does not guarantee a linear quotients ordering of $I(H)^k$, which we provide explicitly here.

Proposition 8.2.1. The k^{th} power $I(H)^k$ of the edge ideal of the antipath H has linear quotients, under the lex ordering of the generators.

We begin with some notation and a lemma.

Given any k edges $e_1, \ldots, e_k$ in a graph G, we will often abuse notation and write $\mathbf{m} = e_1 e_2 \cdots e_k$ for the monomial

$$\mathbf{m} = \prod_{r=1}^{k} x_{i_r} x_{j_r}$$

where $e_r = \{x_{i_r}, x_{j_r}\}$. When a monomial $\mathbf{m}$ is of this form, we say $\mathbf{m}$ is the *product of k edges* of G.

Example 8.2.2. Let G be the complete graph on six vertices $\{x, y, z, w, s, t\}$ seen in Figure 8.2.

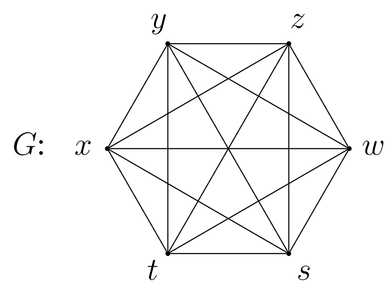


Figure 8.2: G complete graph on 6 vertices.

Then the monomial $\mathbf{m} = xyzwst \in I(G)^3$ comes from any three edges with each vertex appearing in a unique edge exactly once. So $\mathbf{m} = e_1e_2e_3$ for the

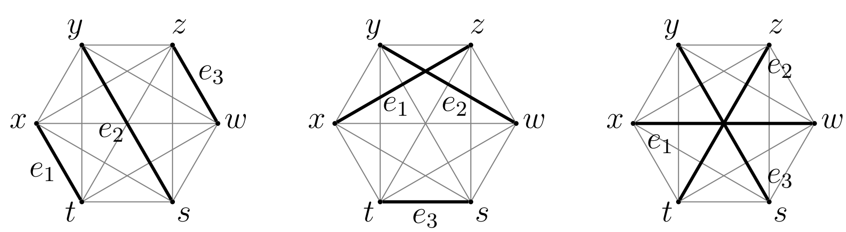


Figure 8.3: Decompositions of $\mathbf{m}$ into three edges.

labeled edge sets in any of the diagrams in Figure 8.3.

Lemma 8.2.3. The ideal $I(H)^k$ is given by all monomials of degree $2k$ of the form

$$I(H)^k = (x_{i_1}x_{i_2}\cdots x_{i_k}x_{j_1}x_{j_2}\cdots x_{j_k} : \\ i_1 \leq i_2 \leq \cdots \leq i_k \leq j_1 \leq j_2 \leq \cdots j_k \text{ and } i_r + 2 \leq j_r \text{ for all } r).$$

Equivalently, every minimal monomial generator $\mathbf{m} \in I(H)^k$ can be written as a product of k edges $\mathbf{m} = e_1 \cdots e_k$ where $e_r = \{x_{i_r}, x_{j_r}\}$ and

$$i_1 \leq i_2 \leq \cdots \leq i_k \leq j_1 \leq j_2 \leq \cdots \leq j_k.$$

Proof. Any monomial $\mathbf{m}$ of degree $2k$ can be written as

$$\mathbf{m} = x_{i_1} \cdots x_{i_k} x_{j_1} \cdots x_{j_k}$$

with $i_1 \leq \cdots \leq i_k \leq j_1 \leq \cdots \leq j_k$. Let $\mathbf{m}$ be a minimal generator of $I(H)^k$ and write $\mathbf{m}$ as above. Assume for a contradiction that there is an index r with $i_r + 2 > j_r$. Since the indices of $\mathbf{m}$ have been written in ascending order, we know that

$$\{i_r, i_{r+1}, \ldots, i_k, j_1, \ldots, j_r\} \subseteq \{i_r, i_r + 1\}.$$

Let $\mathbf{m}'$ be the degree $k+1$ monomial $\mathbf{m}' = x_{i_r} \cdots x_{i_k} x_{j_1} \cdots x_{j_r}$ which divides $\mathbf{m}$. The support of $\mathbf{m}'$ is contained in $\{x_{i_r}, x_{i_r+1}\}$ but there are are no edges in the antipath between x_{i_r} and x_{i_r+1}. Thus, $\mathbf{m}'$ contains no edge as a factor. However, as $\mathbf{m}$ is a product of k edges, every degree $k+1$ factor of $\mathbf{m}$ must contain at least one edge. This is contradicted by our construction of $\mathbf{m}'$, and so we must have $i_r + 2 \leq j_r$ for each r. $\square$

We now return to the proof of Proposition 8.2.1.

Proof of Proposition 8.2.1. From Lemma 8.2.3, we have that

$$\begin{aligned} I(H)^k = (&x_{i_1} x_{i_2} \cdots x_{i_k} x_{j_1} x_{j_2} \cdots x_{j_k} : \\ &i_1 \leq i_2 \leq \cdots \leq i_k \leq j_1 \leq j_2 \leq \cdots j_k \text{ and } i_r + 2 \leq j_r \text{ for all } r). \end{aligned}$$

Any pair of monomial generators $\mathbf{m}$ and $\mathbf{m}'$ of $I(H)^k$ will be of the forms:

$$\mathbf{m} = x_{i_1} x_{i_2} \cdots x_{i_k} x_{j_1} x_{j_2} \cdots x_{j_k} = e_1 e_2 \cdots e_k$$

$$\mathbf{m}' = x_{i'_1} x_{i'_2} \cdots x_{i'_k} x_{j'_1} x_{j'_2} \cdots x_{j'_k} = e'_1 e'_2 \cdots e'_k$$

with indices i_r, i'_r, j_r, j'_r all satisfying the inequalities above and for edges $e_r = \{x_{i_r}, x_{j_r}\}$ and $e'_r = \{x_{i'_r}, x_{j'_r}\}$ of H. We show for every such pair of monomials with $\mathbf{m}' >_{\text{lex}} \mathbf{m}$ that $\mathbf{m}' : \mathbf{m}$ will be divisible by some $x_i = \mathbf{m}'' : \mathbf{m}$ for some $\mathbf{m}'' >_{\text{lex}} \mathbf{m}$.

Case 1: Monomials m and m′ differ first at some x_{i_r}. Assume i_r is the first index at which **m** and **m**′ differ; i.e., $i_s = i'_s$ for all $s < r$ and $i'_r < i_r$.

Let $\mathbf{m}'' = \frac{x_{i'_r}}{x_{i_r}}\mathbf{m}$. This is certainly a monomial of the appropriate degree which is lex earlier than **m**. To show that $\mathbf{m}'' \in I(H)^k$, we note that as $i'_r < i_r < j_r - 2$, we have an edge $\varepsilon_r = \{x_{i'_r}, x_{j_r}\} \in H$. Thus

$$\mathbf{m}'' = e_1 \cdots e_{r-1}\varepsilon_r e_{r+1} \cdots e_k \in I(H)^k.$$

As $\mathbf{m}'' : \mathbf{m} = x_{i'_r}$ and $x_{i'_r}$ divides $\mathbf{m}' : \mathbf{m}$, we either had $\mathbf{m}'' = \mathbf{m}'$ (in which case we satisfy the first condition above) or $\mathbf{m}'' \neq \mathbf{m}'$ and this colon satisfies the second condition above.

Case 2: Monomials m and m′ differ first at some x_{j_r}. Assume that **m** and **m**′ do not differ in the x_{i_s}; i.e., $i_s = i'_s$ for all $s = 1, \ldots, k$. Let j_r be the first index where **m** and **m**′ differ. That is, $j_s = j'_s$ for all $s < r$ and $j'_r < j_r$. So

$$\mathbf{m} = x_{i_1} \cdots x_{i_k} x_{j_1} \cdots x_{j_{r-1}} x_{j_r} x_{j_{r+1}} \cdots x_{j_k} = e_1 e_2 \cdots e_{r-1} e_r e_{r+1} \cdots e_k$$
$$\mathbf{m}' = x_{i_1} \cdots x_{i_k} x_{j_1} \cdots x_{j_{r-1}} x_{j'_r} x_{j'_{r+1}} \cdots x_{j'_k} = e_1 e_2 \cdots e_{r-1} e'_r e'_{r+1} \cdots e'_k.$$

Choosing

$$\begin{aligned}\mathbf{m}'' &= x_{i_1} \cdots x_{i_k} x_{j_1} \cdots x_{j_{r-1}} x_{j'_r} x_{j_{r+1}} \cdots x_{j_k} \\ &= e_1 e_2 \cdots e_{r-1} e'_r e_{r+1} \cdots e_r,\end{aligned}$$

we note that as $e'_r = \{x_{i_r}, x_{j'_r}\} \in H$, we have $\mathbf{m}'' \in I(H)^k$. This is a lex earlier monomial in $I(H)^k$. So $\mathbf{m}'' : \mathbf{m} = x_{j'_r}$ which divides $\mathbf{m}' : \mathbf{m}$. □

8.3 Linear Quotient Ordering of Anticycle

The proof that the square of the edge ideal of the antipath has linear quotients is the first step in constructing a linear quotients ordering of the generators of the anticycle. With this in hand, we now show that the following ordering on the generators of the square of the edge ideal of the anticycle gives us linear quotients. For the remainder of this note, we let G be the anticycle graph and let H be the antipath obtained by deleting some vertex of G.

Remark 8.3.1. We will label the vertices in G as follows. Let x be the vertex we delete to obtain H, and let z_1 and z_2 the two non-adjacent vertices in G (so the two neighbors of x in the cycle itself). Finally, let $y_1, \ldots, y_n$ be all the remainging vertices in order, so that y_1 is not adjacent to z_1 and y_n is not adjacent to z_n. Note that each y_i is adjacent to x. Thus, for this section, we assume that G has $n + 3$ vertices. See the figure below.

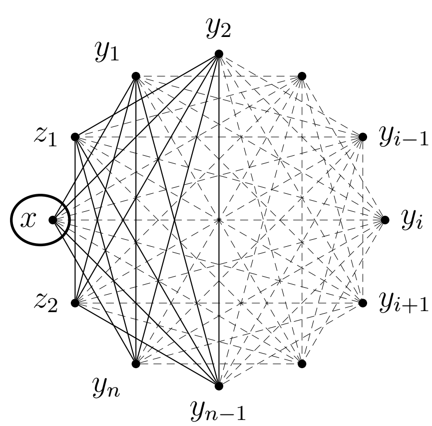


Figure 8.4: Labeled anticycle graph.

Theorem 8.3.2. Let G be the $(n + 3)$-anticycle graph, labeled as in the picture above, with $n \geq 2$. Let $H = G \setminus \{x\}$ be the induced graph away from x. Let

$J = I(H)$ be the edge ideal of H and let $K = I(G \setminus H) = (xy_i : i = 1, \ldots, n)$ be the edge ideal on the edges not in H.

Then the edge ideal $I(G)$ has a linear quotients given by the following ordering of its monoimal generators (monomials occurring earlier in this list appear earlier in the order):

1. $\mathbf{m} \in J^2$ ordered via the lex ordering with $z_1 < y_1 < y_2 < \cdots < y_n < z_2$
2. $\mathbf{m} \in J \cdot K$
   (a) $\mathbf{m} = xy_iz_1z_2$, $i = 1, \ldots, n$,
   (b) $\mathbf{m} = xy_iy_jz_2$, $i \leq j$, ordered via lex with $y_1 > y_2 > \cdots > y_n$, excluding nongenerator $xy_n^2z_2$,
   (c) $\mathbf{m} = xy_iy_jz_1$, $i \leq j$, ordered via lex with $y_1 < y_2 < \cdots < y_n$, excluding nongenerator $xy_1^2z_1$, and
   (d) $\mathbf{m} = xy_iy_jy_k$, $i \leq j \leq k$, ordered via lex with $y_1 > y_2 > \cdots > y_n$.
3. $\mathbf{m} \in K^2$.
   (a) $\mathbf{m} = x^2y_iy_j$ ordered via lex excluding $x^2y_1^2$ with $y_1 < y_2 < \cdots < y_n$
   (b) $\mathbf{m} = x^2y_1^2$.

Before giving the proof, we provide a specific example of the ordering of $I(G)^2$ for the antipath G on 6 vertices.

Example 8.3.3. Let $n = 3$ so we have the anticycle graph G on vertices $\{x, z_1, y_1, y_2, y_3, z_2\}$.

Our two subgraphs H and $G \setminus H$ will be as below. The linear quotients

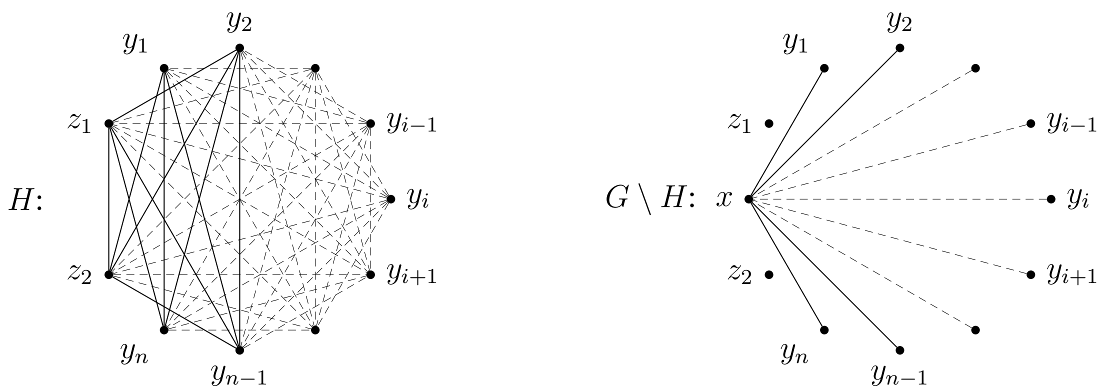


Figure 8.5: Anticycle graph decomposed into H and G/H.

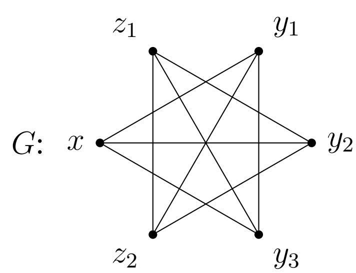


Figure 8.6: Anticycle graph on 6 vertices.

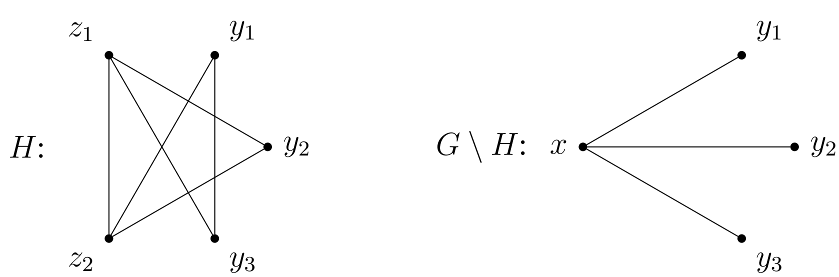


Figure 8.7: Anticycle decomposed into H and $G \setminus H$.

ordering from Theorem 8.3.2 on the generators of $I(G)^2$ is given here by

$$\begin{aligned} I(G)^2 = \quad & (z_1^2y_2^2, z_1^2y_2y_3, z_1^2y_2z_2, z_1^2y_3^2, z_1^2y_3z_2, z_1^2z_2^2, z_1y_1y_2y_3, z_1y_1y_2z_2, \\ & z_1y_1y_3^2, z_1y_1y_3z_2, z_1y_1z_2^2, z_1y_2^2z_2, z_1y_2y_3z_2, z_1y_2z_2^2, y_1^2y_3^2, \\ & y_1^2y_3z_2, y_1^2z_2^2, y_1y_2y_3z_2, y_1y_2z_2^2, y_2^2z_2^2)^{(1)} \\ & + (xz_1y_1z_2, xz_1y_2z_2, xz_1y_3z_2)^{(2a)} \end{aligned}$$

8.3.1 Proof of Theorem 8.3.2

Proof of Theorem 8.3.2. The generators of $I(G)^2$ fall into three main cases, with the second case split up into four subcases and the third case placing the first lex ordered generator at the very end. We will address each case separately.

Notation 8.3.4. Let $I_M = (I(G)^2)_M$ denote the ideal generated by all monomials in the linear quotients ordering before adding M, a minimal generator of $I(G)^2$. In general, we will use Q_M to denote the colon ideal

$$Q_M = I_M : (M),$$

though we will often omit the subscript if the stage in the ordering is clear. We show here for all monomial generators M in the quotients ordering that

$$Q_M = (x_{i_1}, x_{i_2}, \ldots, x_{i_k})$$

for some variables $x_{i_1}, x_{i_2}, \ldots, x_{i_k} \in \{x, z_1, z_2, y_1, y_2, \ldots, y_n\} = V$.

Let V_M denote the variables generating Q_M, or as above, $V_M = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ and let $W_M = V \setminus V_M$.

The general technique used begins with generating $x_i \in V_M$ explicitly via exhibition of a monomial generator $\mathbf{m}' \in I_M$ such that

$$\mathbf{m}' : M = x_i.$$

After finding our expected V_M, we note that any remaining minimal monomial generators $\mathbf{m}$ of Q_M which are not variables, i.e. not in a linear generator of the ideal (V_M), must have their support, $\mathbf{supp}(\mathbf{m}) \in W_M$.

We then show that any generators $\mathbf{m}' \in I(G)^2$ which would give us

$$\mathbf{m}' : M = \mathbf{m} \in (W_M)$$

must either have $\mathbf{m} \in (V_M)$ (and hence a contradiction, as such a generator cannot be minimal in Q_M) or could only come from a monomial $\mathbf{m}'$ occurring after M in the linear quotients ordering (and hence another contradiction, as $\mathbf{m} \notin Q_M$.) For consistency, we will always use M, $\mathbf{m}$ and $\mathbf{m}'$ in the same roles throughout the proof.

Stage (1):

Note that $I(H)$ is the antipath graph of the path $\{z_1 \sim y_1 \sim y_2 \sim \cdots \sim y_n \sim z_2\}$, so the ordering of J^2 given in (1) is a linear quotients ordering by Proposition 8.2.1.

Stage (2a):

We now move on to generators in $(2a)$ and show that after adding through the $(i-1)^{\text{st}}$ term in $(2a)$, we have linear quotients when we colon this ideal against our i^{th} term, $M = z_1 z_2 x y_i$. Let Q be this colon ideal,

$$\begin{aligned} Q &= I_{z_1 z_2 x y_i} : (z_1 z_2 x y_i) \\ &= (J^2 + (z_1 z_2 x y_j \mid 1 \leq j \leq i-1)) : (z_1 z_2 x y_i). \end{aligned}$$

Note that the following inclusions hold, via the elements noted on the right.

- $Q \supseteq (y_j \mid j \neq i)$ as $y_j = z_1 z_2 y_j y_i : z_1 z_2 x y_i$.
- $Q \supseteq (z_1)$ when $i \neq 1$ as $z_1 = z_1 z_2 z_1 y_i : z_1 z_2 x y_i$.
- $Q \supseteq (z_2)$ when $i \neq n$ as $z_2 = z_1 z_2 z_2 y_i : z_1 z_2 x y_i$.
- $Q \supseteq (y_i)$ when $i \notin \{1, n\}$ as $y_i = y_i^2 z_1 z_2 : z_1 z_2 x y_i$.

Assume $\mathbf{m} \in Q$ is a minimal monomial generator of Q that is not linear, i.e. $\mathbf{m} = \mathbf{m}' : z_1z_2xy_i$ for some $\mathbf{m}'$ appearing in the ordering earlier than $z_1z_2xy_i$. As $\mathbf{m}$ is minimal, its support cannot contain any of the variables in Q and therefore

$$\mathbf{supp}(\mathbf{m}) \subseteq \begin{cases} \{x\} & i = 2, \dots, n-1, \\ \{x, z_1, y_1\} & i = 1, \\ \{x, z_2, y_n\} & i = n. \end{cases}$$

In the first of these cases, we note that if $x|\mathbf{m}$ then $x^2|\mathbf{m}'$. As this does not happen for any $\mathbf{m}'$ before $z_1z_2xy_i$, the only cases we need to consider are $i = 1$ and $i = n$. In both of these cases we can assume that x does not divide $\mathbf{m}$.

Case ($i = 1$): In this case, we are adding the generator $z_1z_2xy_1$ to J^2, our edge ideal of the antipath, i.e. $Q = J^2 : z_1z_2xy_1$. Note that $Q \supseteq (y_2, \dots, y_n, z_2)$. Hence, if we have a minimal monomial generator $\mathbf{m} \in Q$ which is not linear, its support must be contained in $\{z_1, y_1\}$.

If $z_1|\mathbf{m}$ then $z_1^2|\mathbf{m}'$ so $\mathbf{m}'$ must be of the form $z_1^2y_jy_k$ with $j, k > 1$. However, we then have $\mathbf{m}' : z_1z_2xy_1 = z_1y_jy_k$ which cannot be a minimal generator of Q, as both $y_j, y_k \in Q$.

If $y_1|\mathbf{m}$ then $y_1^2|\mathbf{m}'$ so $\mathbf{m}'$ must be of the form $y_1^2y_jz_2$ (for $j > 2$) or $y_1^2y_jy_k$ (for $j, k > 2$) or $y_1^2z_2^2$. In these three cases the $\mathbf{m}'$ are y_1y_j, $y_1y_jy_k$, and y_1z_2 respectively. However each of these are not minimal, from $y_j, z_2 \in Q$ for $j > 2$.

Case ($i = n$): Now we are adding the final generator $z_1z_2xy_n$ to the ideal

$$I_{z_1z_2xy_n} = J^2 + (z_1z_2xy_i : 1 \leq i \leq n-1).$$

For this, we have $Q = (J^2 + (z_1z_2xy_j \mid 1 \leq j \leq n-1)) : (z_1z_2xy_n)$ which satisfies $Q \supseteq (y_1, \ldots, y_{n-1}, z_1)$. In this case, if we have a minimal monomial generator $\mathbf{m} \in Q$ which is not linear, its support must be contained in $\{z_2, y_n\}$.

If $z_2|\mathbf{m}$ then $z_2^2|\mathbf{m}'$. The only such $\mathbf{m}' \in I_{z_1z_2zy_n}$ must be of the form $z_2^2y_jy_k$ with $j, k < n$. However, we then have $\mathbf{m}' : z_1z_2xy_1 = z_2y_jy_k$ which is not a minimal generator as $y_j, y_k \in Q$.

Similarly, if $y_n|\mathbf{m}$ then $y_n^2|\mathbf{m}'$. All such $\mathbf{m}' \in I_{z_1z_2zy_n}$ are of one of the following three forms:

(i) $y_n^2y_jz_1$ (for some $j < n-1$)

(ii) $y_n^2y_jy_k$ (for some $j, k < n-1$)

(iii) $y_n^2z_1^2$.

In these three cases the $\mathbf{m} = \mathbf{m}' : M$ is

(i) $\mathbf{m} = y_n^2y_jz_1 : z_1z_2zy_n = y_jy_n$,

(ii) $\mathbf{m} = y_n^2y_jy_k : z_1z_2zy_n = y_jy_ky_n$, and

(iii) $\mathbf{m} = y_n^2z_1^2 : z_1z_2zy_n = y_nz_1$ respectively.

However each of these are not minimal as $y_j, z_1 \in Q$ for $j < n-1$.

So our ordering of our generators is a linear quotients ordering through the end of stage (2a).

Stage (2b):

The second part of the second stage involves adding monomials $M = xy_iy_jz_2$ to our ideals I_M according to the lex order on (i,j).

$$\begin{aligned} Q &= I_{xy_iy_jz_2} : (xy_iy_jz_2) \\ &= (J^2 + (z_1z_2xy_j \mid 1 \le j \le n) + (xy_{i'}y_{j'}z_2 : (i',j') >_{\text{lex}} (i,j)) : (xy_iy_jz_2) \end{aligned}$$

Note the following inclusions hold, via the elements noted.

- $Q \supseteq (y_k \mid k < j)$ as $y_k = xy_iy_kz_2 : xy_iy_jz_2$
- $Q \supseteq (z_1)$ as $z_1 = xy_iz_1z_2 : xy_iy_jz_2$
- $Q \supseteq (z_2)$ when $j \neq n$ as $z_1 = y_iy_jz_2^2 : xy_iy_jz_2$
- $Q \supseteq (y_k \mid k > j+1)$ as $y_k = y_iy_jy_kz_2 : xy_iy_jz_2$
- $Q \supseteq (y_{j+1})$ when $i \neq j$ as $y_{j+1} = y_iy_jy_{j+1}z_2 : xy_iy_jz_2$
- $Q \supseteq (y_j)$ when $i \le j-2$ and $j \neq n$ as $y_j = y_iy_j^2z_2 : xy_iy_jz_2$

Taken together for $M = xy_iy_jz_2$ this gives

$$Q \supseteq \begin{cases} (y_1, \ldots, y_n, z_1, z_2) & j \neq n, i < j-1 \\ (y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n, z_1, z_2) & j \neq n, i+1 = j \\ (y_1, \ldots, y_{j-1}, y_{j+2}, \ldots, y_n, z_1, z_2) & j \neq n, i = j \\ (y_1, \ldots, y_{n-1}, z_1) & j = n. \end{cases}$$

Assume $\mathbf{m} \in Q$ is a minimal monomial generator that is not linear. That is $\mathbf{m} = \mathbf{m}' : xy_iy_jz_2$ for some $\mathbf{m}'$ before $xy_iy_jz_2$. As $\mathbf{m}$ is minimal, its support cannot contain any of the variables in Q. Also if x were to be in $\mathbf{supp}(\mathbf{m})$ then x^2

would divide $\mathbf{m}'$. As no there is no such $\mathbf{m}' \in I_M$ before $xy_iy_kz_2$, we have $x \not| \mathbf{m}$. Thus the support of $\mathbf{m}$ satisfies

$$\mathbf{supp}(\mathbf{m}) \subseteq \begin{cases} \emptyset & j \neq n, i < j-1 \\ \{y_j\} & j \neq n, i+1 = j \\ \{y_j, y_{j+1}\} & j \neq n, i = j \\ \{y_n, z_2\} & j = n \end{cases}$$

Case ($j \neq n, i < j-1$): There is nothing to check as x does not divide $\mathbf{m}$ and all other variables are in Q.

Case ($j \neq n, i+1 = j$): In this case $\mathbf{m}$ must be a power of y_j. As $\mathbf{m}$ is not linear, $y_j^2|\mathbf{m}$ and hence $y_j^3|\mathbf{m}'$. However none of the generators of $I(G)^2$ are divisible by y_j^3.

Case ($j \neq n, i = j$): In this case $\mathbf{supp}(\mathbf{m}) \subseteq \{y_j, y_{j+1}\}$. As $\mathbf{m}$ is not linear, we have one of the following must hold:

(i) $y_j^2|\mathbf{m}$

(ii) $y_jy_{j+1}|\mathbf{m}$

(iii) $y_{j+1}^2|\mathbf{m}$.

In these three cases respectively we must then have

(i) $y_j^3|\mathbf{m}'$

(ii) $y_j^2y_{j+1}|\mathbf{m}'$

(iii) $\mathbf{m}' \in \left\{y_j^2y_{j+1}^2, y_jy_{j+1}^3, y_{j+1}^4, xy_jy_{j+1}^2, xy_{j+1}^3, z_2y_jy_{j+1}^2, z_2y_{j+1}^3, xz_2y_{j+1}^2\right\}$.

Case **(i)** cannot happen, as y_j^3 does not divide any generator of $I(G)^2$. Similarly, in case **(ii)**, $y_j^2y_{j+1}|\mathbf{m}'$ which would require $y_jy_{j+1} \in I(G)$, which is not a generator of the edge ideal of the anticycle.

Finally, in case **(iii)** all degree 4 monomials divisible by y_{j+1}^2 have been enumerated as possible $\mathbf{m}'$. None of these are generators of $I(G)^2$ except for $\mathbf{m}' = xz_2y_{j+1}^2$. This however occurs later in our order.

Case ($j = n$): In this case $\mathbf{supp}(\mathbf{m}) \subseteq \{y_n, z_2\}$. As $\mathbf{m}$ is not linear, one of y_n^2, y_nz_2 and z_2^2 divide $\mathbf{m}$. If y_n^2 or z_2^2 divide $\mathbf{m}$ then y_n^3 or z_2^3 divide $\mathbf{m}'$. However no generator of $I(G)^2$ is divisible by a cube of a variable. If $y_nz_2|\mathbf{m}$ then $\mathbf{m}' = y_n^2z_2^2$ which is not a generator of $I(G)^2$.

Stage (2c):

Showing that this part of the ordering is a linear quotients ordering can be done using its symmetry with Stage (2b). We wish to show that all Q such that

$$\begin{aligned} Q &= I_{xy_iy_jz_1} : (xy_iy_jz_1) \\ &= \Big(J^2 + \big(z_1z_2xy_j \mid 1 \leq j \leq n\big) + \big(xy_ky_lz_2 \mid 1 \leq k \leq l \leq n, k < n\big) \\ &\qquad + \big(xy_ky_lz_1 \mid (k,l) <_{\text{lex}'} (i,j)\big)\Big) : (xy_iy_jz_1) \end{aligned}$$

are again generated by variables. We first show that Q' is generated by variables, for

$$Q' = \Big(J^2 + \big(z_1z_2xy_j \mid 1 \leq j \leq n\big) + \big(xy_ky_lz_1 \mid (k,l) <_{\text{lex}'} (i,j)\big)\Big) : (xy_iy_jz_1),$$

where the $<_{\text{lex}'}$ denotes the lex ordering on y_i with the variables in reverse order from the $<_{\text{lex}}$ used in Stage (2b).

Via symmetry with Stage (2b), this Q' must have linear quotients via an identical proof. From this, we see

$$Q' = \begin{cases} (y_1, \dots, y_n, z_1, z_2) & j \neq n, j < i-1 \\ (y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n, z_1, z_2) & i \neq 1, j+1 = i \\ (y_1, \dots, y_{j-1}, y_{j+2}, \dots, y_n, z_1, z_2) & i \neq 1, i = j \\ (y_2, \dots, y_n, z_2) & i = 1. \end{cases}$$

Clearly $Q' \subset Q$. We note that Q and Q' only differ by a colon ideal of the form

$$\big(xy_ky_lz_2 \mid 1 \leq k \leq l \leq n, k < n\big) : (xy_iy_jz_1).$$

The generators of Q which are not in Q' are of the form $xy_ky_lz_2 : xy_iy_jz_1$ and hence all must divisible by z_2.

Since $z_2 \in Q'$ in all cases, we see that Q is generated by variables for all monomials M added in this stage.

Stage (2d):

For the final case of Stage 2, we add all monomials in $J \cdot K$ of the form $\mathbf{m} = xy_iy_jy_k$ ordered via lex with $y_1 > y_2 > \cdots y_n$. Our colon ideals then are of the form

$$\begin{aligned} Q &= I_{xy_iy_jy_k} : (xy_iy_jy_k) \\ &= \Bigg(J^2 + \big(z_1z_2xy_j \mid 1 \leq j \leq n\big) \\ &+ \big(xy_ky_lz_2 \mid 1 \leq k \leq l \leq n, k < n\big) + \big(xy_ky_lz_1 \mid 1 \leq k \leq l \leq n, 1 < l\big) \\ &+ \big(xy_{i'}y_{j'}y_{k'} \mid 1 \leq i' \leq j' \leq k' \leq n, i'+2 \leq k', (i', j', k') >_{\text{lex}} (i, j, k)\big)\Bigg) : (xy_iy_jy_k). \end{aligned}$$

The last set of generators in $I_{xy_iy_jy_k}$ are given by

$$\left(xy_{i'}y_{j'}y_{k'} \mid 1 \leq i' \leq j' \leq k' \leq n, i'+2 \leq k', (i',j',k') >_{\text{lex}} (i,j,k)\right)$$

as the variables can be arranged with indices i', j', k' in increasing order, but $i'+2 \leq k'$ as at least one pair of $\{y_{i'}, y_{j'}, y_{k'}\}$ must be nonadjacent in the anticycle graph. This forces the given inequality.

Our colon ideals now satisfy the following inclusions, via the elements noted.

- $Q \supseteq (y_l \mid l < j)$ as $y_l = xy_iy_ly_k : xy_iy_jy_k$
- $Q \supseteq (z_2)$ as $z_2 = xy_iy_kz_2 : xy_iy_jy_k$
- $Q \supseteq (z_1)$ as $z_1 = xy_iy_kz_1 : xy_iy_jy_k$
- $Q \supseteq (y_l \mid l \geq j+2)$ as $y_l = y_iy_jy_ky_l : xy_iy_jy_k$
- $Q \supseteq (y_{j+1})$ when $i+1 \leq j$ and $j+2 \leq k$ as $y_{j+1} = y_iy_jy_{j+1}y_k : xy_iy_jy_k$
- $Q \supseteq (y_j)$ when $i+2 \leq j$ and $j+2 \leq k$ as $y_{j+1} = y_iy_j^2y_k : xy_iy_jy_k$.

Together this gives

$$Q \supseteq \begin{cases} (y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n, z_1, z_2) & i = j-1 \text{ and } j+2 \leq k \\ (y_1, \ldots, y_{j-1}, y_{j+2}, \ldots, y_n, z_1, z_2) & i = j \text{ or } j = k, k-1 \\ (y_1, \ldots, y_n, z_1, z_2) & \text{otherwise.} \end{cases}$$

Assume $\mathbf{m} \in Q$ is a minimal monomial generator that is not linear. That is $\mathbf{m} = \mathbf{m}' : xy_iy_jy_k$ for some $\mathbf{m}'$ before $M = xy_iy_jy_k$. As $\mathbf{m}$ is minimal, its support cannot contain any of the variables in Q. Also if $x|\mathbf{m}$ then $x^2|\mathbf{m}'$. As this does not happen for any $\mathbf{m}'$ before $xy_iy_jy_k$, $x \notin \mathbf{supp}(\mathbf{m})$. Thus the support of $\mathbf{m}$ satisfies

$$\mathbf{supp}(\mathbf{m}) \subseteq \begin{cases} \{y_j\} & i = j-1 \text{ and } j+2 \leq k \\ \{y_j, y_{j+1}\} & i = j \text{ or } j = k, k-1 \\ \emptyset & \text{otherwise.} \end{cases}$$

Case ($i = j - 1$ and $j + 2 \leq k$): In this case, **m** must be divisible only by y_j and cannot be linear. Thus $y_j^2|\mathbf{m}$ and $y_j^3|\mathbf{m}'$ which does not hold for any generator $\mathbf{m}' \in I(G)^2$.

Case ($i = j$ or $j = k, k-1$): In this case, **m** has its support contained in $\{y_j, y_{j+1}\}$. As in the previous case, if the support of **m** contains $\{y_j\}$, we obtain a contradiction.

If the support of **m** contains $\{y_{j+1}\}$ and then $\mathbf{m}'$ must the product of y_{j+1}^2 and two of x, y_i, y_j, y_k. However, for this to be a generator of $I(G)^2$ the two chosen vertices must both be adjacent to y_j. If $i = j$, then $\mathbf{m}'xy_{j+1}^2y_k$ is the only possibility, but this comes after $xy_j^2y_k$ in our ordering. If $j = k$ or $j = k-1$ then $\mathbf{m}' = xy_iy_{j+1}^2$ is the only possibility. This again lies after $M = xy_iy_jy_k$ in the ordering.

Other Cases In the other cases, the quotient contains all variables (except x, but there is no term divisible by x^2 which occurs prior to M in the ordering.) Hence, Q must be generated by linear terms.

Stage (3a):

Now we move on to adding those terms in K^2, meaning monomials in $I(G)^2$ which came from pairs of edges xy_i and xy_j. Our colon ideals will be of the form:

$$
\begin{aligned}
Q &= I_{x^2y_iy_j} : (x^2y_iy_j) \\
&= \Big(J^2 + \big(z_1z_2xy_j \mid 1 \le j \le n\big) + \big(xy_ky_lz_2 \mid 1 \le k \le l \le n, k < n\big) \\
&+ \big(xy_ky_lz_1 \mid 1 \le k \le l \le n, 1 < l\big) + \big(xy_iy_jy_k \mid 1 \le i \le j \le k \le n, i+2 \le k\big) \\
&+ \big(x^2y_ky_l \mid 1 \le k \le l \le n, 1 < l, (k,l) >_{\text{lex}} (i,j)\big)\Big) : (x^2y_iy_j).
\end{aligned}
$$

These colon ideals satisfy the following inclusions via the elements noted.

- $Q \supseteq (y_1)$ when $j > 3$ as $y_1 = xy_1y_iy_j : x^2y_iy_j$
- $Q \supseteq (y_1)$ when $i > 1$ as $y_1 = x^2y_1y_i : x^2y_iy_j$
- $Q \supseteq (y_k \mid 1 < k < j)$ as $y_k = x^2y_iy_k : x^2y_iy_j$
- $Q \supseteq (y_k \mid i+2 \le k \le n)$ as $y_k = xy_iy_jy_k : x^2y_iy_j$
- $Q \supseteq (z_2)$ when $i \ne n$ as $z_2 = xy_iy_jz_2 : x^2y_iy_j$
- $Q \supseteq (z_1)$ when $j \ne 1$ as $z_1 = xy_iy_jz_1 : x^2y_iy_j$

Together this gives

$$
Q \supseteq \begin{cases}
(y_3, \dots, y_n, z_1, z_2) & i = 1, j = 2 \\
(y_1, \dots, y_n, z_1, z_2) & i + 2 \le j \\
(y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n, z_1, z_2) & 1 < i = j - 1 \\
(y_1, \dots, y_{j-1}, y_{j+2}, \dots, y_n, z_1, z_2) & 1 < i = j < n \\
(y_1, \dots, y_{n-1}, z_1) & i = j = n.
\end{cases}
$$

Assume $\mathbf{m} \in Q$ is a minimal monomial generator that is not linear. That is $\mathbf{m} = \mathbf{m}' : x^2y_iy_j$ for some $\mathbf{m}'$ before $M = x^2y_iy_j$. Again, as $\mathbf{m}$ is minimal its support cannot contain any of the variables in Q. Also if $x|\mathbf{m}$ then $x^3|\mathbf{m}'$ which does not happen for any $\mathbf{m}' \in I(G)^2$. Thus the support of $\mathbf{m}$ satisfies

$$\mathbf{supp}(\mathbf{m}) \subseteq \begin{cases} \{y_1, y_2\} & i = 1, j = 2 \\ \emptyset & i + 2 \leq j \\ \{y_j\} & i = j - 1 \\ \{y_j, y_{j+1}\} & 1 < i = j < n \\ \{y_n, z_2\} & i = j = n. \end{cases}$$

We examine each of these cases individually.

Case ($i = 1, j = 2$): In this case $\mathbf{m}$ is divisible by one of y_1^2, y_1y_2, y_2^2 and hence $\mathbf{m}'$ is divisible by $y_1^3, y_1^2y_2^2, y_2^3$. None of these can hold for $\mathbf{m}'$ a generator of $I(G)^2$.

Case ($i + 2 \leq j$): There is nothing to check as x does not divide $\mathbf{m}'$ and all other variables are in Q.

Case ($i = j - 1$): In this case $\mathbf{m}$ must be a power of y_j. As $\mathbf{m}$ is not linear, $y_j^2|\mathbf{m}'$ and hence $y_j^3|\mathbf{m}$. No generators of $I(G)^2$ are divisible by y_j^3 (or any third power of a variable.)

Case ($1 < i = j < n$): In this case $\mathbf{m}$ is divisible by one of y_j^2, y_jy_{j+1} or y_{j+1}^2. If $\mathbf{m}'$ is to appear before $x^2y_iy_j$ in our list, it cannot be $x^2y_j^2, x^2y_jy_{j+1},$ nor $x^2y_{j+1}^2$. As $i = j$, the remaining possibilities for $\mathbf{m}$ are $xy_j^3, xy_j^2y_{j+1}, xy_jy_{j+1}^2$ or a monomial of degree four in y_j and y_{j+1}. However, none of these are generators of $I(G)^2$.

Case ($i = j = n$): In this case $\mathbf{m}$ is divisible by one of y_n^2, y_nz_2, z_2^2. So $\mathbf{m}'$ is divisible by one of y_n^4, $y_n^3z_2$, z_2^2. There are no $\mathbf{m}' \in I(G)^2$ such that the first two hold. For the last, if $z_2^2|\mathbf{m}'$ and y_n does not divide $\mathbf{m}$ then $\mathbf{m}'$ must be one of $z_2^4, z_2^3x, z_2^3y_1, z_2^2x^2, z_2^2xy_n, z_2^2y_n^2$. None of these are in $I(G)^2$.

From this, we see that $I(G)^2$ has a linear quotients through Stage (3a).

Stage (3b):

Finally, we add our generator $x^2y_1^2$ to our ideal $I_{x^2y_1^2}$. We only need to check that for this one remaining generator, the following colon ideal is generated by variables:

$$\begin{aligned}
Q &= I_{x^2y_1^2} : (x^2y_1^2) \\
&= \Big(J^2 + \big(z_1z_2xy_j \mid 1 \leq j \leq n\big) + \big(xy_ky_lz_2 \mid 1 \leq k \leq l \leq n, k < n\big) \\
&+ \big(xy_ky_lz_1 \mid 1 \leq k \leq l \leq n, 1 < l\big) + \big(xy_iy_jy_k \mid 1 \leq i \leq j \leq k \leq n, i+2 \leq k\big) \\
&+ \big(x^2y_ky_l \mid 1 \leq k \leq l \leq n, 1 < l\big)\Big) : (x^2y_1^2).
\end{aligned}$$

We have the following inclusions by the elements noted:

- $Q \supseteq (y_k \mid 1 < k \leq n)$ as $y_k = x^2y_1y_k : x^2y_1^2$
- $Q \supseteq (z_2)$ when $i \neq n$ as $z_2 = xy_1^2z_2 : x^2y_1^2$.

This gives us that our colon ideal satisfies $Q \supseteq (y_2, \ldots, y_n, z_2)$.

So, if $\mathbf{m} \in Q$ is a minimal non-linear monomial, then $\mathbf{supp}(\mathbf{m}) \subseteq \{y_1, z_1\}$ and $\mathbf{m} = \mathbf{m}' : x^2y_1^2$ for some $\mathbf{m} \in I(G)^2$ before $x^2y_1^2$. If $y_1|\mathbf{m}$ then $\mathbf{m}'$ must be divisible by y_1^3. There is no such $\mathbf{m}' \in I(G)^2$. Thus $\mathbf{supp}(\mathbf{m}) = \{z_1\}$.

Since by assumption, $\mathbf{m}$ is not linear, $z_1^2|\mathbf{m}$. Thus, $z_1^2|\mathbf{m}'$ and the other variables dividing $\mathbf{m}'$ can only be z_1, x or y_1. There is no way to form a generator of

$I(G)^2$ using only these variables as y_1 and x and z_1 are not adjacent to z_1^2. Hence, $Q = (y_2, \ldots, y_n, z_2)$.

So this provides a linear quotients ordering on $I(G)^2$. $\square$

8.4 Future Research

For higher powers of the edge ideal $I(A_n)^k$ of the anticycle, it is still unknown if all powers have a linear resolution, much less linear quotients. Construction of linear quotient orderings on $I(A_n)^k$ would accomplish this.

Question 8.4.1. Does $I(A_n)^k$ have linear quotients for $k \geq 3$?

We produced an ordering above on $I(A_n)^2$ by decomposing A_n into complementary subgraphs P_{n-1} and $A_n \setminus P_{n-1}$. While this order is nonunique, ordering the edges of $I(A_n)^2$ by decomposing the graph into the complementary subgraphs H and $G \setminus H$, then considering pairs of edges as appropriate, seems to produce linear quotients orderings with the clearest descriptions. Extending this order to $I(G)^k$ in a similar fashion has proven fairly difficult, even in the case of $I(G)^3$, but would be a natural next step after Theorem 8.3.2.

A problem of more general interest is to complete Theorem 8.0.6 of Herzog, Hibi and Zheng by answering the following question:

Question 8.4.2. Let G be the complement of a chordal graph. Does $I(G)^k$ have linear quotients for $k \geq 2$?

We might also ask for a description of all edge ideals whose powers eventually have linear resolutions.

Question 8.4.3. Can we exhibit classes of graphs G such that for all sufficiently large k,

(i) $I(G)^k$ has a linear resolution, or

(ii) $I(G)^k$ has linear quotients?

In [29], it was conjectured that graphs satistfying Question 8.4.3(i) are precisely those graphs G with a C_4-free complement. General conditions for the second class however remain open. It appears that anticycles A_n form such a class, but we wish to find more general conditions for the powers of an edge ideal of a graph to stabilize on linear quotients.

CHAPTER 9

NERVE COMPLEXES OF GRAPHS

Most algorithms for the enumeration of spanning trees of a graph involve construction of a particular spanning tree, then creating a computation tree describing a pattern of edge swaps in such a way as to eventually list all possible spanning trees of the graph. In this thesis, we describe squarefree monomial ideals whose multigraded Betti numbers naturally enumerate spanning trees (along with several other invariants.)

Throughout, all graphs are assumed to be simple (meaning no loops or multiple edges), undirected, and unlabeled. The field $\mathbb{k}$ that all polynomials are taken over is fixed throughout, but no assumptions need to be made about it (characteristic, algebraic completeness, etc.) For computational purposes, resolving these ideals over $\mathbb{Z}_2$ is sufficient.

The relevant simplicial complex is the nerve (or neighborhood) complex of a graph G. As the nerve of a subgraph of G will have the same homology type as the subgraph itself, we can convert homological calculations for subgraphs of G into homological calculations for induced subcomplexes of $\mathcal{N}(G)$. For purposes of computing spanning trees, it is sufficient to calculate the resolution of the Stanley-Reisner ideal of $\mathcal{N}(G)$ only through the $n-3$ stage to enumerate all spanning trees, where $n = |V|$.

Any graph invariants of G which depend on homologies of subgraphs can be read off from the resolution of $\mathcal{N}(G)$. For example, the genus of the graph G is given by another Betti number - specifically, the last Betti number in the nonlinear strand of the resolution. Other invariants include matchings of size k of G, maximum or minimum degrees of vertices in G, the Tutte polynomial, the k-edge connectivity, the lengths of minimal cycles, the number (or existence) of Hamiltonian cycles, and others.

The Betti numbers of the Alexander duals of these complexes enumerate other graph invariants, as well as providing an interesting parallel to the reconstruction conjecture.

9.1 Nerve Theorem, Neighborhood Complexes and Subgraphs

It is sufficient to consider a very specific type of *nerve complex* (rather than general nerves) for our purposes here.

Definition 9.1.1. Let Δ be a simplicial complex, $\mathcal{F}_\Delta$ the facets of Δ. The *nerve* $\mathcal{N}(\Delta)$ *of* Δ is the simplicial complex on vertex set $\mathcal{F}_\Delta$ with a face $\sigma = \{F_{i_1}, ..., F_{i_k}\}$ whenever $F_{i_1} \cap \cdots \cap F_{i_k} \neq \emptyset$.

Remark 9.1.2. We compute this for graphs as dimension 2 simplicial complexes with the edges as facets. For graphs, we will also call $\mathcal{N}(G)$ the *neighborhood complex* of G.

Example 9.1.3. If Δ is a simplicial complex with 2-dimensional facets as in Figure 9.1, then $\mathcal{N}(\Delta)$ is a tetrahedron, and $\mathcal{N}(\mathcal{N}(\Delta))$ is a point.

Example 9.1.4. The nerves of graphs arise naturally in other situations. For example, on n vertices, we have that the complete graph has the complex of claw graphs on n vertices as its nerve, i.e. $\mathcal{N}(K_n) \cong \Delta(\mathcal{F}_{claws}[n])$. See Figure 9.2 for an example.

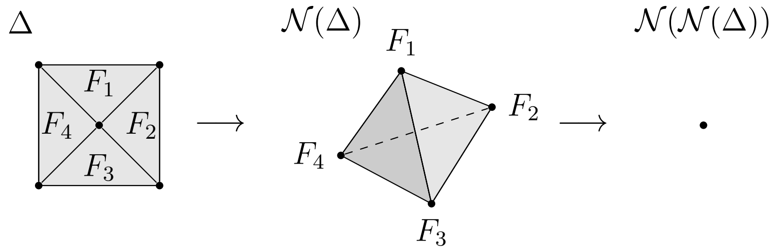


Figure 9.1: Nerve of a Nerve of a Complex: $\mathcal{N}(\mathcal{N}(\Delta))$

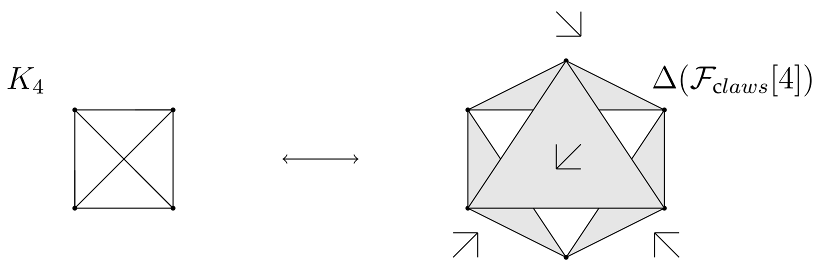


Figure 9.2: Complete Graph K_4 and Simplicial Complex of Claw Graphs

This provides an alternate way of calculating the homotopy type of $\Delta(\mathcal{F}_{claws}[n])$. For a more in depth look at simplicial complexes of graphs, see [23].

Viewing $\mathcal{N}$ as a function from the set of finite simplicial complexes to itself, we note that $\mathcal{N}$ rarely has $\mathcal{N}(\mathcal{N}(\Delta)) = \Delta$. The class of complexes which satisfy $\mathcal{N}(\mathcal{N}(\Delta)) = \Delta$ are called *taut complexes.* A general study of the nerve complexes $\mathcal{N}(\Delta)$ can be found in [14]. Characterization of the graphs which give rise to taut complexes is simpler than in the general case. This is given in Proposition 9.1.8.

We note a few properties of general $\mathcal{N}(\Delta)$.

Definition 9.1.5. We say that a vertex v is a *leaf of a simplicial complex* Δ if v lies in a unique facet $F \in \mathcal{F}_\Delta$, where $\mathcal{F}_\Delta$ denotes the set of maximal facets of Δ.

Proposition 9.1.6. Let Δ be a simplicial complex and $\mathcal{N}(\Delta)$ be its nerve complex. We have the following:

(i) The vertices of $\mathcal{N}(\Delta)$ are in 1:1 correspondence with the facets of Δ.

(ii) The facets of $\mathcal{N}(\Delta)$ are in 1:1 correspondence with non-empty intersections of facets $\sigma_1 \cap \cdots \cap \sigma_k \in \Delta$ of maximal size, i.e. all sets $\mathcal{S} = \{\sigma_1, ..., \sigma_k\}$ such that

$$\cap\mathcal{S} = \sigma_1 \cap \cdots \cap \sigma_k \neq \emptyset$$

but $\cap\mathcal{S} \cap \tau = \emptyset$ for all facets $\tau \notin \mathcal{S}$.

Proof of Proposition 9.1.6. The proposition follows immediately from the definition. □

Corollary 9.1.7. For a graph G and neighborhood complex $\mathcal{N}(G)$, we have:

(i) The vertices of $\mathcal{N}(G)$ are in 1:1 correspondence with the edges of G, i.e.

$$V_{\mathcal{N}(G)} \longleftrightarrow E_G.$$

(ii) The facets of $\mathcal{N}(G)$ are in 1:1 correspondence with the non-leaf vertices of G, i.e.

$$\begin{aligned} \mathcal{F}_{\mathcal{N}(\Delta)} &\longleftrightarrow \{v \in \Delta : \text{v non-leaf vertex}\} \\ &\longleftrightarrow \{v \in G : \deg(v) > 1\}. \end{aligned}$$

(iii) Leaf vertices of $\mathcal{N}(G)$ correspond precisely to edges $e = \{v, w\}$ such that v or w is a leaf of G.

Proof of Corollary 9.1.7. The proof of 9.1.7 is immediate from Proposition 9.1.6. We need only to show that non-empty intersections of maximal size in G correspond to non-leaf vertices. We note that the faces of $\mathcal{N}(\Delta)$ are of the form

$$\sigma = \{e_{i_1}, e_{i_2}, ..., e_{i_k}\}$$

with $e_{i_1} \cap e_{i_2} \cap \cdots \cap e_{i_k} \neq \emptyset$, and any set of distinct edges which mutually intersect nontrivially must contain a common vertex v. Any maximal such face must include all edges adjacent to v. We denote the face corresponding to all edges incident to a vertex v by σ_v.

A vertex v of degree 1 will have exactly one adjacent edge $e = \{v, w\}$ for some $w \in V$. The set of edges containing w is a facet of $\mathcal{N}(G)$ containing e, so a vertex of degree 1 does not contribute a maximal facet, as $\sigma_v \subseteq \sigma_w$. As v is a vertex of degree 1, no other vertices w' are adjacent to v, so e is a leaf of $\mathcal{N}(G)$, contained only in facet σ_w, showing statement 9.1.7(iii).

Otherwise, if v and w are vertices both of degree higher than 1, they have at most one edge shared between v and w and at least one edge containing v but not w and one edge containing w but not v between them. Hence, $\sigma_v \not\subseteq \sigma_w$ and $\sigma_w \not\subseteq \sigma_v$.

So we have a distinct maximal facet σ_v for every vertex $v \in G$ of degree greater than 1, i.e.

$$\mathcal{F}_{\mathcal{N}(G)} = \{\sigma_v : v \in G, \deg v > 1\}.$$

□

Proposition 9.1.8. The nerve of the nerve of a graph G is G if and only if G is a leaf-free graph.

Proof. We have that the facets σ_v of $\mathcal{N}(G)$ are in 1:1 correspondence with the non-leaf vertices of G. So $\mathcal{N}(\mathcal{N}(G))$ will have vertex set

$$V_{\mathcal{N}(\mathcal{N}(G))} = \{v \in G : \deg v > 1\}.$$

Hence, $V_{\mathcal{N}(\mathcal{N}(G))} \subsetneq V_G$ unless G is a leaf-free graph.

Restricting now to the case G a leaf-free graph, we show that $\mathcal{N}(\mathcal{N}(G)) = G$. The facets of $\mathcal{N}(G)$ correspond to vertices of G, with two facets σ_v and σ_w intersecting nontrivially if and only if $e = \{v, w\} \in E_G$. So $\mathcal{N}(\mathcal{N}(G))$ has vertex set $V_{\mathcal{N}(\mathcal{N}(G))} = V_G$ and faces f_e for each edge $e \in E_G$. So

$$\mathcal{F}_{\mathcal{N}(\mathcal{N}(G))} = E_G,$$

and we have $G = \mathcal{N}(\mathcal{N}(G))$. $\square$

We add one final lemma which we will use in the Morse theoretic proof that $\mathcal{N}(G) \cong G$.

Lemma 9.1.9. Any two facets σ_v and σ_w in $\mathcal{N}(G)$ intersect in at most one vertex $e \in V_{\mathcal{N}(G)}$.

Proof. This is clear, as two facets intersect precisely when the vertices they correspond to have an edge $e = \{v, w\}$ between them in E_G. $\square$

Example 9.1.10 (Running Example). The graph in Figure 9.3 is a leaf-free graph, with $\mathcal{N}(G) \cong G$.

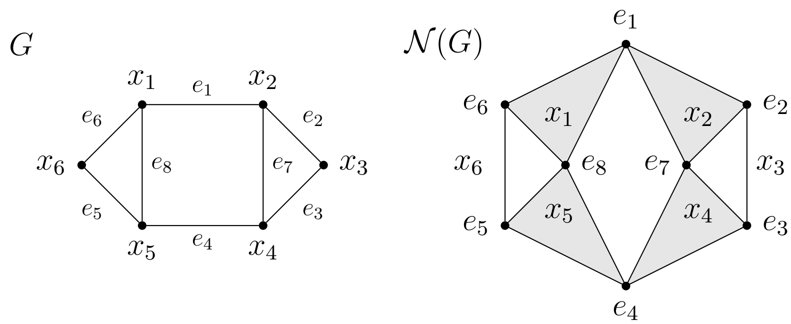


Figure 9.3: Graph G with Neighborhood Complex $\mathcal{N}(G)$

9.2 Properties of Nerve Complexes

The Nerve theorem gives both that $\mathcal{N}(G) \cong G$, and that $G[W] \cong \mathcal{N}(G[W])$ for any subgraph (V, W) of (V, E). We include a direct proof of this homotopy equivalence via stellar subdivision and discrete Morse theoretic arguments in Theorem 9.2.2.

Theorem 9.2.1 (Nerve Theorem, [14]). Let K be a simplicial complex with facets $F_1, ..., F_n$. Then

$$\mathcal{N}(K) = \mathcal{N}(\{F_1, F_2, ..., F_n\}) \cong ||K||,$$

i.e. the nerve is homotopy equivalent to the geometric realization of K.

We provide an explicit proof here that $\mathcal{N}(G) \cong G$ for graphs G.

Theorem 9.2.2 (Nerve Theorem for Neighborhood Complexes). Let G be a graph on vertex set $V = \{v_1, ..., v_n\}$ with edge set $E = \{e_1, ..., e_m\}$. Then if $W = \{e_{i_1}, ..., e_{i_k}\} \subseteq E$ is a subset of edges of E, and $G[W]$ the subgraph on the induced vertex set, we have

$$\mathcal{N}(G[W]) = \mathcal{N}(\{e_{i_1}, ..., e_{i_k}\}) \cong G[W],$$

where $G[W]$ is viewed as a simplicial complex.

Proof of Theorem 9.2.2. It is sufficient to prove that our graph $G \cong \mathcal{N}(G)$, as construction of the nerve of $G[W]$ relies only on data in the edge induced subgraph. So without loss of generality, we assume that $W = E$. Let st(Δ) denote the stellar subdivision of a simplicial complex Δ. We construct a chain of homotopy equivalences of the form

$$G \cong \text{st}(G) \cong \text{st}(\mathcal{N}(G)) \cong \mathcal{N}(G),$$

which will give our desired homotopy equivalence.

That $G \cong \text{st}(G)$ and $\mathcal{N}(G) \cong \text{st}(\mathcal{N}(G))$ follow from stellar subdivision preserving homotopy types. We construct a Morse matching on the faces of st($\mathcal{N}(G)$) which contracts st($\mathcal{N}(G)$) onto st(G) via simplicial collapses.

The stellar subdivision of a simplicial complex Δ replaces each facet $F \in \mathcal{F}_\Delta$, $|F| = d+1$, with a new central vertex v_F surrounded by $d+1$ new d-dim'l faces of the form

$$\sigma \cup \{v_F\}$$

where $\sigma \in \partial F$. So for facet $F = \{v_{i_0}, v_{i_1}, ..., v_{i_d}\}$, the facets of st($\Delta$) are given by

$$\mathcal{F}_{\text{st}(\Delta)} = \left\{ \{v_{i_0}, ..., \widehat{v_{i_j}}, ..., v_{i_d}\} \cup \{v_F\} \quad : \quad j = 0, ..., d \right\}.$$

From Corollary 9.1.7, we have the vertices of $\mathcal{N}(G)$ in 1:1 correspondence with the edges of G, the facets of $\mathcal{N}(G)$ are in 1:1 correspondence with non-leaf vertices of G. Combining this, we see that the vertex set of the stellar subdivision of $\mathcal{N}(G)$ is

$$V_{\text{st}(\mathcal{N}(G))} = \{e : e \in E_G\} \cup \{v : v \in V_G, \deg v > 1\},$$

where we abuse the notation somewhat and refer to these added vertices in the stellar subdivision of $\mathcal{N}(G)$ by their labels $v \in G$.

Let σ_v denote the facet of $\mathcal{N}(G)$ corresponding to a vertex $v \in G$ with $\deg v > 1$. The Morse matching M on st($\mathcal{N}(G)$) is given by

$$M = \left\{ \big(\sigma \backslash \{v\}, \sigma\big) : \sigma \subset \text{st}(\sigma_v), |\sigma| > 2 \right\}$$

for each non-leaf $v \in V_G$ and each face σ not a vertex or edge of st($\mathcal{N}(G)$).

We check our two conditions for this to be a Morse matching inside the subposet $P_{\text{st}(\sigma_v)}$ of the inclusion poset $P_{\mathcal{N}(G)}$, as all arcs begin and end in faces inside some fixed st(σ_v).

The condition that all faces $\tau \in \sigma_v$ be contained in at most one endpoint of an arc in M is clear, as if $v \in \tau$ and $|\tau| > 2$, then $(\tau \backslash \{v\}, \tau)$ is the unique arc in M containing τ as an endpoint. Similarly, if $v \notin \tau$, $|\tau| > 1$, then $(\tau, \tau \cap \{v\})$ is the unique arc containing τ as an endpoint. If τ satisfies neither of these conditions, then τ remains unmatched in M. Acyclicity is clear as well, as within $P_{\text{st}(\sigma_v)}$, we are always adding and removing the same vertex v.

Our critical cells of M are then all faces τ not of the form described above. Specifically,

$$\begin{aligned} C_M = & \{\{v\} : v \in V_G, \deg v > 1\} \\ & \cup \{\{e\} : e \in E_G\} \\ & \cup \big\{\{v, e\} : v \in e \text{ in } G\big\}. \end{aligned}$$

Let $\overline{\text{st}(\mathcal{N}(G))}$ denote the contraction to the cells C_M via the Morse matching above. This is almost the stellar subdivision of G. The only places where st(G) and $\overline{\text{st}(\mathcal{N}(G))}$ differ are at the stellar subdivision of the leaves of G.

To remedy this, we let $\overline{\text{st}(G)}$ denote the graph st(G) with the leaves after the stellar subdivision contracted. This graph is homotopy equivalent to st(G) and is the same graph as $\overline{\text{st}(\mathcal{N}(G))}$, completing the proof. □

Example 9.2.3 (Running Example). In Figure 9.4, we slightly modify our running example to include a leaf vertex. Note the chain of homotopy equivalences in Theorem 9.2.2 end in a common complex Δ_G.

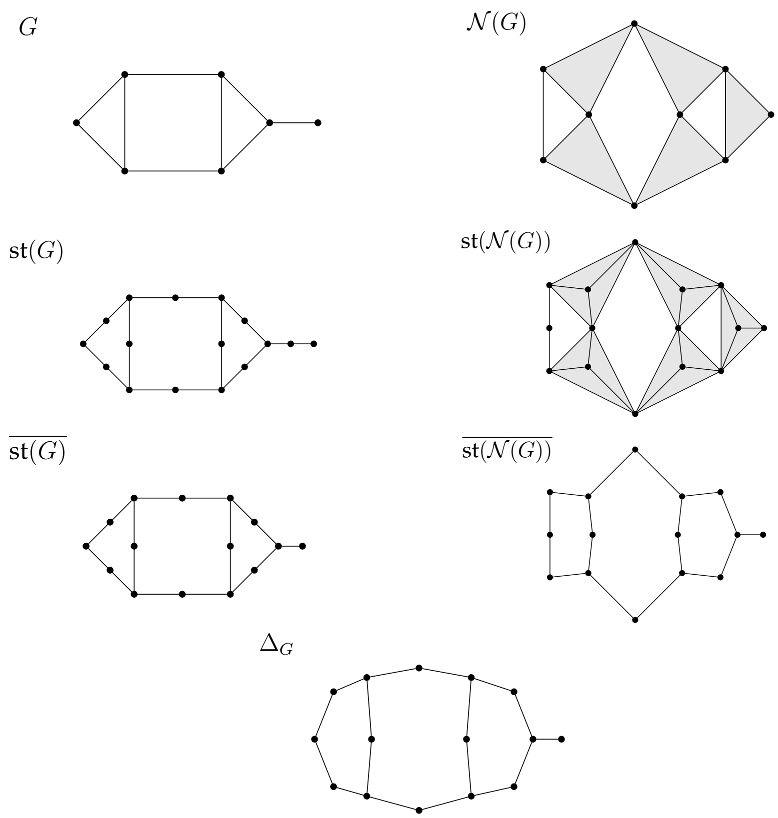


Figure 9.4: Series of Subdivisions and Contractions giving $\mathcal{N}(G) \cong G$.

9.2.1 Regularity of Ideals of Nerve Complexes of Graphs

We set $\mathbf{m}_W = \prod_{w \in W} x_w$ for each $W \subset E$.

Theorem 9.2.4 bounds the regularity of $\Bbbk[\mathcal{N}(G)]$, as all induced subcomplexes have $\widetilde{H}_i(G[W]) = \widetilde{H}_i(\mathcal{N}(G)|_W) = 0$ for $i > 2$. The two nonzero strands of the resolution of $\Bbbk[\mathcal{N}(G)]$ read off homological invariants of subgraphs of G via the following theorem.

Theorem 9.2.4. Let $\Bbbk[\mathcal{N}(G)]$ denote the Stanley Reisner ring of $\mathcal{N}(G)$, a quotient of the polynomial ring $R = \Bbbk[e_1, ..., e_m]$. Let $\beta_{i,\mathbf{m}_W}(\Bbbk[\mathcal{N}(G)])$ denote the multigraded Betti number of $\Bbbk[\mathcal{N}(G)]$ at homological stage i in multidegree $\mathbf{m}_W$. Let $|\mathbf{m}_W|$ denote $\deg(\mathbf{m}_W) = |W|$. Then

$$\beta_{|\mathbf{m}_W|-1,\mathbf{m}_W}(\Bbbk[\mathcal{N}(G)]) = \dim \widetilde{H}_0(G[W]) = \#\{\text{connected components of } G[W]\} - 1$$
$$\beta_{|\mathbf{m}_W|-2,\mathbf{m}_W}(\Bbbk[\mathcal{N}(G)]) = \dim \widetilde{H}_1(G[W]) = \#\{\text{loops in } G[W]\},$$

where $G[W]$ denotes the subgraph of G on edges W.

Proof. This is immediate from $\mathcal{N}(G)|_W \cong G[W]$ via Theorem 9.2.2 and Hochster's formula. $\square$

This will be our primary computational tool when investigating invariants of G on subgraphs. We will demonstrate in Section 9.3 some uses of Theorem 9.2.4 in rewrite graph calculations in terms of algebraic invariants of $\Bbbk[\mathcal{N}(G)]$.

Corollary 9.2.5. Let G be a graph, $\mathcal{N}(G)$ its neighborhood complex. Then

(i) $\operatorname{reg}(I_{\mathcal{N}(G)}) \leq 3$

(ii) $\operatorname{reg}(I_{\mathcal{N}(G)}) = 2 \iff G$ is a tree.

Proof of Corollary 9.2.5. As $\mathcal{N}(G) \cong G$, we have that $\widetilde{H}_j(\mathcal{N}(G)) = 0$ for all $i \geq 2$. So by Hochster's formula,

$$\beta_{i,i+j}(\Bbbk[\mathcal{N}(G)]) = 0$$

for $j \geq 2$. Hence, $\text{reg}(I_{\mathcal{N}(G)}) \leq 3$.

The nonlinear strand is zero precisely when all subgraphs of $G[W]$ are loop-free, which is the case only if G is a tree. $\square$

The generating monomials for $I_{\mathcal{N}(G)}$ can be recovered from this formula as the multigraded Betti numbers in homological degree 1 of total degrees 2 and 3.

Corollary 9.2.6. Let G be a graph, $\mathcal{N}(G)$ its neighborhood complex. Then

$$I_{\mathcal{N}(G)} = (e_ie_j : e_i \cap e_j = \emptyset) + (e_ie_je_k : \{e_i, e_j, e_k\} \text{ 3-cycle in G})$$

Proof of Corollary 9.2.6. $\beta_{1,e_ie_j}(\Bbbk[\mathcal{N}(G)])$ is nonzero precisely when $e_i \cap e_j = \emptyset$, i.e. when e_i and e_j are disjoint in G and hence have no face between them in $\mathcal{N}(G)$. Similarly, $\beta_{1,e_ie_je_k}(\Bbbk[\mathcal{N}(G)])$ is nonzero precisely when edges e_i, e_j, e_k form a 3-cycle in G. As $\text{reg}(I_{\mathcal{N}(G)}) \leq 3$, this must give our entire generating set of $I_{\mathcal{N}(G)}$. $\square$

9.3 Graph Invariants and Betti Numbers of Nerve Complexes of Graphs

Any homological invariant of subgraphs can be recovered from the graded or multigraded Betti numbers, including (but not limited to) the spanning trees of

G, the genus of G, the edge connectivity number and the Tutte polynomial. We include here a partial list with proof.

9.3.1 Spanning Trees and Genus of G

Code to enumerate spanning trees of a graph G using this algorithm is available in the *SpanningTrees.m2* package for Macaulay 2. This code is included in Appendix B.

Theorem 9.3.1 (Enumeration of Spanning Trees)**.** Let G be a graph on vertex set $\{x_1, ..., x_n\}$ with edges $\{e_1, ..., e_k\}$. Then the set of spanning trees $T(G)$ of G is given by

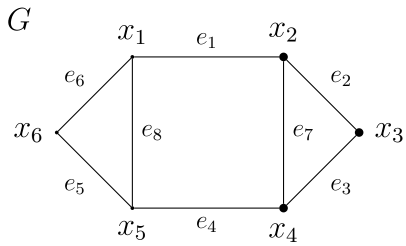
$$T(G) = \left\{ \{e_{i_1}, ..., e_{i_{n-1}}\} : \beta_{n-3,\mathbf{m}_W}(\mathbb{k}[\mathcal{N}(G)]) = 0, \quad W = \{e_{i_1}, ..., e_{i_{n-1}}\} \right\}.$$

Proof. By Theorem 9.2.4, we have that

$$\beta_{|\mathbf{m}_w|-2,\mathbf{m}_W}(\mathbb{k}[\mathcal{N}(G)]) = \#\{\text{loops in } G[W]\}.$$

We consider all squarefree monomials of degree $n-1$, then select those which have no loops. This will be precisely the set of spanning trees of G, proving the theorem. □

Example 9.3.2 (Running Example). Returning to our earlier example with graph G, we have a resolution with the Betti diagram in Figure 9.5.



-	0	1	2	3	4	5	6
total:	1	16	45	57	41	17	3
0:	1	·	·	·	·	·	·
1:	·	14	34	31	10	1	·
2:	·	2	11	26	31	16	3

Figure 9.5: Graph G and the Betti diagram of the Resolution of $\Bbbk[\mathcal{N}(G)]$

There is a nonzero multigraded Betti number $\beta_{3,\mathbf{m}}(\mathcal{N}(G))$ for each $\mathbf{m} \in M_{3,5}$:

$$\begin{aligned}M_{3,5} =\{&e_2e_3e_4e_5e_7, e_2e_3e_4e_6e_7, e_1e_2e_3e_4e_5, e_1e_2e_3e_4e_6, e_1e_4e_5e_7e_8,\\ &e_2e_4e_5e_7e_8, e_3e_5e_6e_7e_8, e_1e_3e_5e_6e_8, e_1e_2e_5e_7e_8, e_1e_2e_3e_4e_7,\\ &e_1e_2e_4e_5e_6, e_3e_4e_5e_7e_8, e_2e_3e_4e_5e_8, e_2e_3e_4e_6e_8, e_1e_3e_5e_7e_8,\\ &e_2e_3e_5e_7e_8, e_1e_3e_6e_7e_8, e_1e_3e_4e_5e_6, e_2e_3e_4e_5e_6, e_4e_5e_6e_7e_8,\\ &e_1e_3e_5e_6e_7, e_2e_3e_4e_7e_8, e_1e_2e_3e_5e_6, e_1e_2e_3e_4e_8, e_1e_5e_6e_7e_8,\\ &e_2e_5e_6e_7e_8\}\end{aligned}$$

So the set of Betti numbers $\beta_{3,\mathbf{m}}(\mathcal{N}(G)) = 0$ corresponds to the set of remaining subgraphs of G of size 5. So $T(G)$ consists of the $\binom{8}{5} - 26 = 30$ subgraphs in Figure 9.6.

In this example,

$$\begin{aligned}\#\{\text{spanning subtrees}\} &= \binom{8}{5} - \beta_{3,5}\\ &= 56 - 26\\ &= 30.\end{aligned}$$

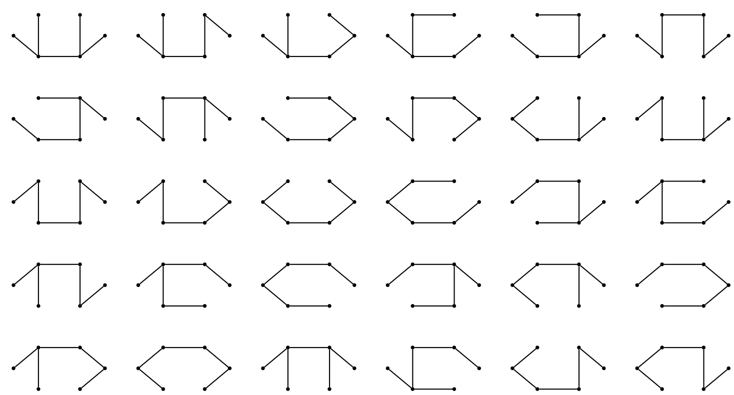


Figure 9.6: Spanning trees of G

In general, however, the Betti number will overcount the number of nonspanning subgraphs (weighted by the number of loops in that subgraph.) This forces us to use multigraded Betti numbers, rather than just the graded.

To compute the genus of the graph, we note that the final Betti number in the nonlinear strand of the resolution of $I_{\mathcal{N}(G)}$ counts the number of loops in G. This gives us the following formula for genus in terms of a singly graded Betti number $\beta_{i,j}(\Bbbk[\mathcal{N}(G)])$:

Proposition 9.3.3 (Genus of G)**.** Let G be a graph on vertex set $V = \{x_1, ..., x_n\}$ with edge set $E = \{e_1, ..., e_k\}$. The genus $g(G)$ of G is given by

$$g(G) = \beta_{k-2,k}(\Bbbk[\mathcal{N}(G)]).$$

Proof of Proposition 9.3.3. From the remark above and from Theorem 9.2.4, the theorem is clear. □

9.3.2 Minimal Cycles and Hamiltonian Cycles

Proposition 9.3.4 (Minimal Cycles $\text{MinCycle}_k(G)$ of length k)**.** Let G be a graph on n vertices and let $\mathcal{N}(G)$ be its neighborhood complex. Let V_W denote the vertex set of a set of edges $W = \{e_{i_1}, ..., e_{i_k}\}$ and let

$$\text{MinCycle}_k(G) = \{W = \{e_{i_1}, ..., e_{i_k}\} : G|_{V_W} \text{ is a cycle of length } k\}$$

be the set of cycles in G of length k. Then $\text{MinCycle}_k(G)$ is the set of all subgraphs (V, W) of G on edge sets $W = \{e_{i_1}, ..., e_{i_k}\}$ such that,

(i) $\beta_{k-2,\mathbf{m}_W}(\Bbbk[\mathcal{N}(G)]) = 1$

(ii) $\beta_{|\mathbf{m}'|-2,\mathbf{m}'}(\Bbbk[\mathcal{N}(G)]) = 0$ for all $\mathbf{m}'|\mathbf{m}_W$, $\mathbf{m}' \neq \mathbf{m}_W$.

Proof of Proposition 9.3.4. If $W = \{e_{i_1}, ..., e_{i_k}\}$ is a minimal cycle if and only if it has $\widetilde{H}_1(G[W]) = 1$ and all $W' \subset W$ have $\widetilde{H}_1(G[W']) = 0$. By Theorem 9.2.4, this is equivalent to statements (i) and (ii) above. □

The existence of a Hamiltonian cycle is equivalent to $\text{MinCycle}_n(G) \neq 0$. Hence, as $\text{MinCycle}_n(G)$ for $n = |V|$ can be computed via the multigraded Betti numbers of the resolution of $\mathcal{N}(G)$, the problem of computing such Betti numbers is NP-hard.

9.3.3 Regularity of G, Minimal/Maximal Vertex Degree

We calculate the maximal and minimal degrees of vertices of graph G via $\Bbbk[\mathcal{N}(G)]$.

Definition 9.3.5. Let G be a graph on vertex set V with edge set E. The *degree of a vertex* v is

$$\deg(v) = \#\{w : \{v,w\} \in E\}.$$

The *maximal degree* M_G *of* G is

$$M_G = \max_v\{\deg(v)\}$$

and the *minimal degree* m_G *of* G is

$$m_G = \min_v\{\deg(v)\}$$

Proposition 9.3.6. Let G be a graph on vertex set $V = \{v_1, ..., v_n\}$ with edge set $E = \{e_1, ..., e_k\}$. Then

(i) $M_G = \max_i\{\deg(v_i)\} = \max_{F_i \in \mathcal{F}} \dim F_i + 1 = \dim \Bbbk[\mathcal{N}(G)] + 1$

(ii)
$$m_G = \min_i\{\deg(v_i)\} = \begin{cases} \min_{F_i \in \mathcal{F}} \dim F_i + 1 & \text{if } G \text{ leaf-free or} \\ 1 & \text{if } G \text{ has a leaf.} \end{cases}$$

where $\mathcal{F}$ is the set of facets of $\mathcal{N}(G)$.

Proof of Proposition 9.3.6. This follows from Corollary 9.1.7 and our construction of $\mathcal{N}(G)$. The facets F_v of $\mathcal{N}(G)$ correspond to the set of all edges incident to a vertex v, and their dimension is $\deg(v) - 1$. □

9.3.4 Matching Number of G

Recall the following definition.

Definition 9.3.7. Let G be a graph on vertex set V with edge set E. Then a *matching of size k of G* is a set of edges $W = \{e_{i_1}, ..., e_{i_k}\} \subseteq E$ such that $e_i \cap e_j = \emptyset$ for all $e_i, e_j \in W$. Let M_k denote the set of all such matchings of size k.

Proposition 9.3.8 (Matchings of Size k in G)**.** Let G be a graph and $\mathcal{N}(G)$ its neighborhood complex. Then

$$M_k = \big\{W = \{e_{i_1}, ..., e_{i_k}\} : \beta_{k-1,\mathbf{m}_W}(\Bbbk[\mathcal{N}(G)]) = k-1\big\}.$$

Proof of Proposition 9.3.8. A matching $W = \{e_{i_1}, ..., e_{i_k}\}$ in M_k will have $\widetilde{H}_0(G[W]) = k-1$. So by Theorem 9.2.4, a set W is a matching of size k if and only if

$$\beta_{k-1,\mathbf{m}_W}(\Bbbk[\mathcal{N}(G)]) = \widetilde{H}_0(G[W]) = k-1,$$

completing the proof. □

The set of induced matchings of G can be enumerated via multigraded Betti numbers of the edge ideal of G via a theorem for hypergraphs specialized here to the case of graphs.

Theorem 9.3.9 (Theorem 6.5 in [16])**.** Let G be a graph. Then $\beta_{i,2i}(R/I_G)$ equals the number of induced matchings of size i of G, where I_G is the edge ideal of G.

9.3.5 Connectedness and k-Edge-Connectivity

We compute connectedness of G and edge-connectivity of G via the linear strand of the resolution of $\Bbbk[\mathcal{N}(G)]$.

Proposition 9.3.10 (Connectedness of G)**.** Let G be a graph on n vertices with edge set E of size k, and let $\mathcal{N}(G)$ be its neighborhood complex. Then G is connected if and only if $\beta_{k-1,k}(\Bbbk[\mathcal{N}(G)]) = 0$.

Proof of Proposition 9.3.10. Using Theorem 9.2.4, we have

$$\beta_{|\mathbf{m}|-1,\mathbf{m}}(\Bbbk[\mathcal{N}(G)]) = \dim \widetilde{H}_0(G[W]) = \#\{\text{connected components of } G[W]\} - 1.$$

So for $W = E$, this becomes

$$\beta_{k-1,\mathbf{m}_E}(\Bbbk[\mathcal{N}(G)]) = \#\{\text{connected components of } G\} - 1 = 0$$

if and only if G is connected. □

Recall the following definition:

Definition 9.3.11. For $k \geq 1$, a graph G is k-edge-connected if G is connected after the removal of any edge set of i edges for $i = 0, \dots, k-1$.

Proposition 9.3.12 (k-Edge-Connectivity of G)**.** Let G be a graph on n vertices with edge set E of size r, and let $\mathcal{N}(G)$ be its neighborhood complex. Let $k \leq n$. Then G is k-edge-connected if and only if $\beta_{r-k-1,r-k}(\Bbbk[\mathcal{N}(G)]) = 0$.

Proof of Proposition 9.3.10. As the linear strand of an edge ideal is zero after the first location where $\beta_{i-1,i}(I_G) = 0$, it suffices to show that $\beta_{r-k-1,k-1}(I_G) = 0$ if and only if I_G is connected after the removal of any edge set of size k.

Again using Theorem 9.2.4, we have

$$\beta_{|\mathbf{m}|-1,\mathbf{m}}(\Bbbk[\mathcal{N}(G)]) = \dim \widetilde{H}_0(G[W]) = \#\{\text{connected components of } G[W]\} - 1.$$

So for W any edge set of size $r-k$, this becomes

$$\beta_{r-k-1,\mathbf{m}_E}(\Bbbk[\mathcal{N}(G)]) = \#\{\text{connected components of } G\} - 1 = 0$$

if and only if (V, W), the subgraph of G obtained after the removal of all edges not in W, is connected. $\square$

9.3.6 Tutte Polynomials

Definition 9.3.13 ([35])**.** For an undirected graph G on vertex set V with edge set E, we define the *Tutte polynomial of* G to be

$$T_G(x,y) = \sum_{W\subset E}(x-1)^{k(W)-k(E)}(y-1)^{k(W)+|W|-|V|},$$

where $k(W)$ denotes the number of connected components of graph (V, W).

Proposition 9.3.14 (Tutte polynomial of G via $\mathcal{N}(G)$)**.** Let G be a graph on vertex set $V = \{x_1, ..., x_n\}$ with edge set $E = \{e_1, ..., e_k\}$. Then the Tutte polynomial is given by

$$T_G(x,y) = \sum_{W\subseteq E}(x-1)^{\left[\beta_{|W|-1,\mathbf{m}_W}(\Bbbk[\mathcal{N}(G)])-\beta_{k-1,\mathbf{m}_E}(\Bbbk[\mathcal{N}(G)])\right]} \cdot (y-1)^{\left[\beta_{|W|-1,\mathbf{m}_W}(\Bbbk[\mathcal{N}(G)])+|W|-n+1\right]}.$$

Note that $\mathbf{m}_E = e_1 e_2 \cdots e_k$.

Proof of Proposition 9.3.14. We rewrite

$$T_G(x,y) = \sum_{W\subset E}(x-1)^{k(W)-k(E)}(y-1)^{k(W)+|W|-|V|}$$

in terms of the Betti numbers of $\Bbbk[\mathcal{N}(G)]$. By Theorem 9.2.4, we have

$$k(W) = \beta_{|W|-1,\mathbf{m}_W}(\Bbbk[\mathcal{N}(G)]) + 1$$

for all subsets $W \subseteq E$. Therefore,

$$
\begin{aligned}
T_G(x,y) &= \sum_{W \subset E} (x-1)^{k(W)-k(E)} (y-1)^{k(W)+|W|-|V|} \\
&= \sum_{W \subseteq E} (x-1)^{\left[\left[\beta_{|W|-1,\mathbf{m}_W}(\Bbbk[\mathcal{N}(G)])+1\right]-\left[\beta_{k-1,\mathbf{m}_E}(\Bbbk[\mathcal{N}(G)])+1\right]\right]} \\
&\qquad\qquad\qquad \cdot (y-1)^{\left[\left[\beta_{|W|-1,\mathbf{m}_W}(\Bbbk[\mathcal{N}(G)])+1\right]+|W|-n\right]} \\
&= \sum_{W \subseteq E} (x-1)^{\left[\beta_{|W|-1,\mathbf{m}_W}(\Bbbk[\mathcal{N}(G)])-\beta_{k-1,\mathbf{m}_E}(\Bbbk[\mathcal{N}(G)])\right]} \\
&\qquad\qquad \cdot (y-1)^{\left[\beta_{|W|-1,\mathbf{m}_W}(\Bbbk[\mathcal{N}(G)])+|W|-n+1\right]},
\end{aligned}
$$

completing the proof. $\square$

APPENDIX A

C_4-FREE COMPLEMENT CODE

```
needsPackage "Nauty"
needsPackage "EdgeIdeals"

buildFiles = (nvertices, nperfile) -> (
    -- place files: graphs-n-1, graphs-n-2, ...
    -- returns number of files created
    time G := generateGraphs(nvertices, OnlyConnected=>true,
    MaxDegree=>nvertices-6);
    nfiles := ceiling(#G / (nperfile * 1.0));
    for i from 1 to nfiles do (
        F := openOut("graphs-"|nvertices|"-"|i);
        for j from nperfile * (i-1) to min(#G-1, nperfile * i  - 1) do
            F << G#j << endl;
        close F;
        );
    nfiles
    );

filter13 = (GG,R) -> select(GG, g -> (
    --filters graphs generated above C_4-free complement
    --then by nonlinear resolutions.
    I := stringToEdgeIdeal(g,R);
    zI := syz gens I;
    maxd := max flatten degrees source zI;
    result := maxd === 3 and regularity I > 2;
    if result then (
```

```
        << betti res I << endl;
        );
        result
    ));

filterGraphFile = (nvertices, fileindex) -> (
    R := ZZ/101[vars(0..nvertices-1)];
    filename := "graphs-"|nvertices|"-"|fileindex;
    G := lines get filename;
    G1 := filter13(G, R);
    F := openOut(filename | ".filter");
    for g in G1 do F << g << endl;
    close F;
    #G1
    )

--List of graphs produced by above code in Graph6 string format:

GG={"J?AFvrw^Fo?","J?BEFo}}@{?","J?B@xzw}Fo?",
    "J?B@xzw}Dw?","J?B@~rw}Fo?","J?B@|zw}Fo?",
    "J?B@|zw}Dw?","J?BfFBwFvo?", "J?bFbx{}@{?",
    "J?rFf_{NFo?","J?rFf_{n@{?"}

--Method to print simplicial complexes in Gap format:
sctoGap(SimplicialComplex,String):=String=>(D,s)->(
    vertset:=gens ring D;
    numvert:=toList(1..#vertset);
    replacepairs:=hashTable apply(numvert, i-> vertset_(i-1) => i);
    protofacetlist:=apply(flatten entries facets D, f-> support f);
```

```
facetlist:=toString apply(protofacetlist,
          f-> apply(f, i-> replacepairs#i));
concatenate(s,":=SCFromFacets(",replace("\\}","]",
          replace("\\{","[",facetlist)),");")
)
```

APPENDIX B

SPANNING TREE CODE

This code, available as an M2 package *SpanningTrees*, will enumerate the spanning trees of a graph G using Macaulay 2 and the M2 package *EdgeIdeals*.

```
--We define first a method for producing the nerve complex
--of a graph, hypergraph or simplicial complex.

nerveComplex = method();

--This simplicial complex is on Vertices = {Edges of G},
--with a facet for every vertex in G:
--This will break as written if any vertices are isolated.
nerveComplex(Graph) := (G) -> (
     m := # edges G;
     kk := coefficientRing ring G;
     S := kk[(symbol e)_1..(symbol e)_m, MonomialSize => 8,
               Degrees => apply(m, i-> apply(m,
               j -> if i === j then 1 else 0))];
     I := apply(vertices G, v -> select(0..(m-1),
               i -> member(v, (edges G)#i)));
     simplicialComplex apply(I,
               L -> product toList apply(L, i-> e_(i+1)))
     )

nerveComplex(HyperGraph):=(H)-> (
```

```
    m := # edges H;
    kk := coefficientRing ring H;
    S := kk[(symbol e)_1..(symbol e)_m, MonomialSize => 8,
              Degrees => apply(m, i-> apply(m,
              j -> if i === j then 1 else 0))];
    I := apply(vertices H, v -> select(0..(m-1),
              i -> member(v, (edges H)#i)));
    simplicialComplex apply(I,
              L -> product toList apply(L, i-> e_(i+1)))
    )

nerveComplex(SimplicialComplex):=(D)->(
    m := # flatten entries facets D;
    kk := coefficientRing ring D;
    S := kk[(symbol e)_1..(symbol e)_m, MonomialSize => 8,
              Degrees => apply(m, i-> apply(m,
              j -> if i === j then 1 else 0))];
    I := apply(gens ring D, v -> select(0..(m-1),
              i -> member(v, support (flatten entries facets D)#i)));
    simplicialComplex apply(I,
              L -> product toList apply(L, i-> e_(i+1)))
    )


--cutRes will truncate resolutions at the relevant multidegree:
cutRes = method();
```

```
cutRes(Graph) := (G) -> (
     g = nerveComplex G;
     time betti res(ideal g,LengthLimit=>(#vertices G-3))
     )



cutRes(Graph,ZZ) := (G,N) -> (
     g := nerveComplex G;
     time betti res(ideal g,LengthLimit=>N)
     )


cutRes(Ideal,ZZ) := (I,N) -> (
     R := ring I;
     betti res(ideal I, LengthLimit=>N)
     )


cutRes(MonomialIdeal,ZZ) := (I,N) -> (
     betti res(ideal I,LengthLimit=> N)
     )


--The following function pulls out non spanning trees of G,
--via zero Betti numbers of the resolution of Nerve(G):


nonSpanningTrees = (G) -> (
     g := nerveComplex G;
```

```
    I := ideal nerveComplex G;
    B := time cutRes G;
    << B << endl << endl;
    threestrandkeys := select(keys B,
              k -> first k === n-3 and last k === n-1);
    threemultidegrees := apply(threestrandkeys,
              k -> k#1);
    threedegindices := apply(threemultidegrees,
              D->select(toList(0..((# edges G)-1)),i->D#i===1));
    apply(threedegindices,
              D-> apply(D,i->product (edges G)#i))
    )

--This function sorts through all possible subsets
--and picks out those which are not "nonSpanningTrees",
--using helper function "sortMinus" below to speed up
-- how M2 looks through the list of Betti numbers.

spanningTrees = (G) -> (
    S := subsets(apply(#edges G,
         i-> product (edges G)#i),(# vertices G)-1);
    T := nonSpanningTrees(G);
    sortMinus(S,T)
    )

sortMinus = (L,M) -> (
```

```
L = sort L;
M = sort M;
i := 0;
j := 0;
done := false;
while i < #L list (
     while not done do (
          while j < #M and M#j < L#i do j = j + 1;
          if j === #M then (
               done = true;
          ) else if M#j == L#i then (
          i = i + 1;
          if i === #L then done = true;
          ) else (
          done = true;
          );
     );
     done = false;
     i = i + 1;
     if i =!= #L + 1 then L#(i-1) else continue
))
```

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